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Being Conservative with a Limited Number of Test Results

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In aircraft structural design, failure stresses are obtained from coupon tests and then used to predict failure under combined loads in structural elements. Structural element tests are next used to update the failure envelope for combined loads. It is a common practice to repeat the element tests and then select the lowest test result as a conservative estimate of the mean failure stress. This practice is equivalent to reducing the average test failure stress by a knockdown factor (one that is quite variable). Instead, we propose using the average test result with an explicit knockdown factor obtained from statistical distribution of the test data. We show reductions in the variability of the estimated mean failure stress as well as the likelihood of unconservative estimate. In addition, when the initial distribution or confidence interval of the mean failure stresses is available, we can further decrease the chance of unconservative estimate using Bayesian updating. We demonstrate the gains associated with Bayesian updating when the upper and lower bounds of errors in the analytical predictions are available. Examples with uniform and lognormal distributions of failure stresses compare the lowest-result approach with the two alternatives with the explicit knockdown factor. Both approaches significantly reduce the likelihood of unconservative estimates of the mean failure stress. The average approach reduced this likelihood by about a half and the Bayesian approach by up to an order of magnitude (from 12.5 to 1%). We also examine scenarios in which estimates of error and variability are substantially inaccurate. We show that, even then, the likelihood of unconservative estimates reduces significantly. Remarkably, an underestimate of variability also results in about a 2% higher average of the estimated mean failure stress. Thus, we are able to simultaneously use higher average failure stress (leading to lower weight) and reduce the likelihood of unconservative estimates.

Nomenclature

b_e	=	error bounds in calculated failure stress
c_f	=	coefficient of variation of the material property (lognormal)
e_f	=	error of calculated failure stress with respect to the average true failure stress
k_{avg}	=	explicit knockdown factor used in the average approach
k_{Bayes}	=	explicit knockdown factor used in the Bayesian approach
k_{lowest}	=	implicit knockdown factor introduced by using the lowest test result as a conservative measure
$\bar{k}_{lowest}$	=	mean value of the implicit knockdown factors
v_f	=	variability in material properties
(σ_f)	=	failure stress (a random variable)
$(\bar{\sigma}_f)_{Bayes}$	=	mean failure stress calculated by Bayesian updating
$(\sigma_f)_{calc}$	=	failure stress predicted from fracture mechanics
$(\sigma_f)_{est}$	=	estimated failure stress for the three discussed methods (a conservative estimate)
$(\sigma_f)_{i,test}$	=	ith test result.
$(\sigma_f)_{test}$	=	failure stress measured in tests
$(\sigma_f)_{true}$	=	true failure stress of a structure
$(\bar{\sigma}_f)_{true}$	=	mean of true failure stresses of an infinite number of nominally identical structures

I. Introduction

AEROSPACE structures have traditionally been designed using a deterministic approach based on the regulations of the Federal Aviation Administration (FAA) or other guidelines. In this approach, safety is achieved by combining safety factors with tests of materials and structural components. On the other hand, there is growing interest in replacing safety-factor-based deterministic design with reliability-based design (e.g., Lincoln [1], Wirsching [2], [3], and Long and Narciso [4]). In the traditional deterministic design approach, the probabilistic concept has already taken place as the choice of allowable design. This is based on statistical analysis of coupon tests, with A-basis or B-basis allowable design selected to be below 99 or 90% of the material populations, respectively, with 95% confidence. That is, the stress allowable (A or B basis) is obtained from the average coupon failure stress times a knockdown factor obtained from statistical analysis.

Motivated by the remarkable safety level of current aircraft, we analyzed the effects of measures that improve aircraft structural safety and compared the relative effectiveness of safety measures taken during aircraft structural design in our earlier work (Acar et al. [5,6]). The safety measures that we included in those works were the load safety factors of 1.5, conservative material properties (A basis or B basis), redundancy, certification (or proof) test, and error and variability reduction. The most common form of error reduction is conducting structural element tests, which is the main focus of the present work. Structural element tests are usually used to conservatively select failure stress by taking the lowest result of a

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batch of nominally identical tests. This process constitutes introducing an implicit *knockdown factor* (a term introduced for correcting analytical predictions based on test results [7]). However, variability in material and test conditions leads to excessive volatility in the knockdown factor and the resulting estimate of mean failure stress. The main objective of the present paper is to show that simple statistical analysis can be used to replace the implicit knockdown factors with explicit ones and reduce the volatility in the estimated mean failure stress. This can be done using the average test result instead of the lowest one. Further improvements may be obtained by using the degree of confidence in the analytical estimates of the mean failure stress via Bayesian updating.

The present paper is organized as follows: Section II provides the analysis of the three options for using element test results to update the estimate of the failure envelope. Section III provides an illustrative example comparing the three approaches for one case of three-element test results. Section IV compares the three approaches for several examples. Section V shows how these results can be directly applied to saving weight. Section VI discusses the uncertainties involved in estimating variability. Section VII presents concluding remarks.

II. Analysis

In the building-block approach of aircraft structural elements, the number of tests is usually limited to three, due to the large number of structural components to test. Consider a typical situation related to updating analytical predictions of strength based on three test data on structural components. First, a failure envelope is constructed using results from coupon tests and a failure criterion such as Tsai–Wu theory [8]. Then nominally identical copies of the structural components are loaded to failure under different load combinations, and the observed failure stresses are used to update the failure envelope, as shown in Fig. 1. A common approach is to shrink the failure envelope by the lowest ratio between the test results and the

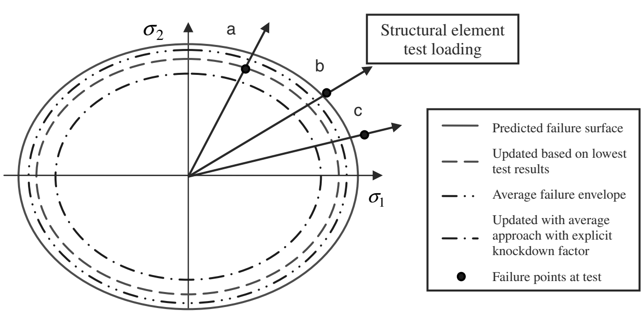


Fig. 1 Failure surface updated by tests. The failure envelope is usually updated based on lowest test results, but it may be more logical to update it based on average result multiplied by an additional knockdown factor.

predicted failure stresses. This can be viewed as applying an implicit knockdown factor to the average test failure stress. If the tests are repeated with a different material sample, the predicted failure stress will be changed as well as the implicit knockdown factor. We argue that it may be better to use the average ratio and then apply an explicit knockdown factor to update the failure envelope. However, the predicted failure stress from failure theory only plays a minor role in both the lowest and average approaches. As the accuracy of analytical failure theories increases, there is a chance that both the lowest and average approaches may provide a less accurate estimate of the true mean failure stress than the failure theory. This is because material and test variability can distort the test results. We will further discuss using a Bayesian technique to take advantage of confidence in the predicted failure stress and combine it with the test data.

We assume that the analytical prediction of failure stress $(\sigma_f)_{\mathrm{calc}}$ applies to the mean failure stress $(\bar{\sigma}_f)_{\mathrm{true}}$ of an infinite number of nominally identical structures; $(\sigma_f)_{\mathrm{calc}}$ is shown by the original failure envelope (solid line) in Fig. 1. The error e_f of our analytical prediction is defined by Eq. (1):

$$(\bar{\sigma}_f)_{\mathrm{true}} = (1 + e_f)(\sigma_f)_{\mathrm{calc}} \tag{1}$$

Here, we assume that the designer can estimate the bounds b_e (possibly conservative) on the magnitude of the error, and we further assume that the errors have a uniform distribution between the bounds:

$$f(e_f) = \begin{cases} \frac{1}{2b_e} & \text{if } |e_f| \le b_e \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

That is, although the error has only one value in reality, the distribution reflects the uncertainty associated with our lack of knowledge of what this value is (so-called epistemic uncertainty). For simplicity, we neglect errors and variability from testing procedures and measurements, which is essentially equivalent to the case of $(\sigma_f)_{\mathrm{test}} = (\sigma_f)_{\mathrm{true}}$. In addition, we simplify the analysis by assuming that the structural elements are all tested under identical conditions, so that the lines a, b, and c in Fig. 1 coincide. The difference in test results reflects only variability in the material properties of the nominally identical structural elements. In that case, the effect of variability in material properties is reflected in Eq. (3):

$$(\sigma_f)_{\mathrm{test}} = (1 + v_f)(\bar{\sigma}_f)_{\mathrm{true}} = (1 + v_f)(1 + e_f)(\sigma_f)_{\mathrm{calc}} \tag{3}$$

Unlike the error, the variability does not reflect lack of knowledge, but the randomness (so-called aleatory uncertainty) introduced into the mean failure stress by the manufacturing process for the material and the structure. Usually, a calculated failure stress comes with bounds of error, and variability is estimated by coupon-test results. This is depicted schematically in Fig. 2 with a uniformly distributed error and a lognormally distributed variability. The distribution of the variability is given as

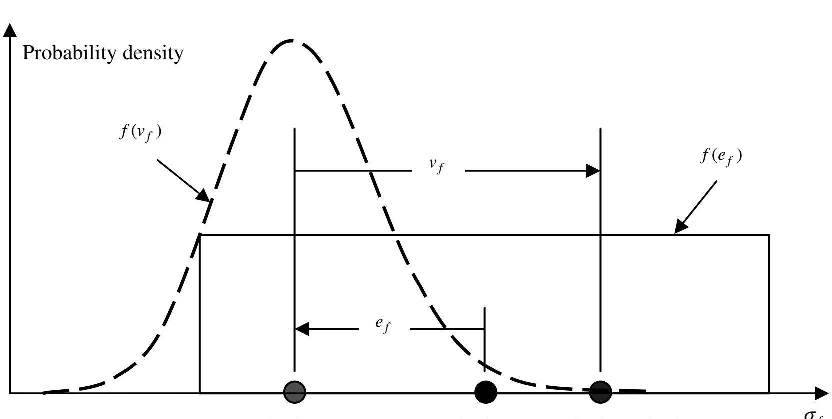


Fig. 2 Error and variability in failure stress. The error is centered around the predicted value, and is assumed to be uniformly distributed. The variability distribution, on the other hand, is lognormal with a mean equal to the true mean failure stress.

$$f(v_f) = LN(v_f + 1; \ln(1), \varsigma_f), \qquad \varsigma_f = \sqrt{\ln(1 + c_f^2)} \tag{4}$$

where c_f is the coefficient of variation (c.o.v.) of the material property.

With an infinite number of tests, the average of test results would be $(\bar{\sigma_f})_{\text{true}}$, and we could calculate e_f exactly from Eq. (1). With a finite number of tests, we can obtain only an estimate of the error, and designers often opt for a conservative estimate.

We now consider three ways of estimating conservative $(\bar{\sigma_f})_{\text{true}}$:

1) The first approach takes the lowest test result $(\bar{\sigma_f})_{\text{est}} = \min(\sigma_f)_{i,\text{test}}$ as a conservative estimate of $(\bar{\sigma_f})_{\text{true}}$, where $(\sigma_f)_{i,\text{test}}$ is the result from the ith test. This is often the approach taken by conservative designers and is equivalent to applying an implicit knockdown factor k_{lowest}. As the lowest test data follow the extreme value distribution [9], the mean value of k_{lowest}, $\bar{k}_{\text{lowest}}$, may be calculated as the average value of the distribution. The derivation and explanation of explicit knockdown factors can be found in the work of Acar et al. [10].

2) The second approach takes the average value of test results and applies an explicit knockdown factor $k_{\text{avg.}}$, and the conservative mean failure stress is estimated by

$$(\bar{\sigma}_f)_{\text{est}} = k_{\text{avg}}(\bar{\sigma}_f)_{\text{test}} \tag{5}$$

In the first set of results, we assume that the explicit knockdown factor is the same as $\bar{k}_{\text{lowest}}$ to achieve the same average estimate of the mean failure stress over all possible tests obtained in the first approach. This should result in the same mean weight of the component.

3) The third approach takes advantage of knowledge of a confidence interval of analytical prediction. Because the error in Eq. (2) is assumed to be uniformly distributed, we generate the initial probability distribution $f^{\text{ini}}(\bar{\sigma}_f)$ of the mean failure stress as

$$f^{\text{ini}}(\bar{\sigma}_f) = \begin{cases} \frac{1}{2b_e(\sigma_f)_{\text{calc}}} & \text{if } \left|\frac{\bar{\sigma}_f}{(\sigma_f)_{\text{calc}}} - 1\right| \le b_e \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Now the initial distribution of the mean failure stress is updated using the Bayesian technique with a given $(\sigma_f)_{1,\text{test}}$ as

$$f^{\text{upd}}(\bar{\sigma}_f) = \frac{f_{1,\text{test}}(\bar{\sigma}_f) f^{\text{ini}}(\bar{\sigma}_f)}{\int_{-\infty}^{\infty} f_{1,\text{test}}(\bar{\sigma}_f) f^{\text{ini}}(\bar{\sigma}_f) \mathrm{d}\bar{\sigma}_f} \tag{7}$$

where $f_{1,\text{test}}(\bar{\sigma_f}) = LN((\sigma_f)_{1,\text{test}}; \bar{\sigma_f}, \varsigma_f)$ is the likelihood function reflecting possible variability of the first test result $(\sigma_f)_{1,\text{test}}$. Note that $f_{1,\text{test}}(\bar{\sigma_f})$ is not a probability distribution in $\bar{\sigma_f}$; it is the conditional probability density of obtaining test result $(\sigma_f)_{1,\text{test}}$, given that the mean value of the failure stress is $\bar{\sigma_f}$. Subsequent tests are handled by the same equations, using the updated distribution, as the initial one. The advantage of the Bayesian updating approach is that it takes into account our confidence in the analytical prediction. If the error bounds are very large, we can expect that this approach would reduce to taking the average of the tests, as we will see in Sec. IV. If the error bounds are very narrow, this approach will not allow the test results to much change the calculated stress, as these test results may reflect variability rather than error.

The average value of the last updated distribution is the Bayesian estimate of the mean failure stress $(\bar{\sigma_f})_{\text{Bayes}}$. Finally, we multiply by an explicit knockdown factor k_{Bayes}; that is,

$$(\bar{\sigma}_f)_{\text{est}} = k_{\text{Bayes}}(\bar{\sigma}_f)_{\text{Bayes}} \tag{8}$$

In our first set of results, the explicit knockdown factor k_{Bayes} of the Bayesian estimate has the same value with $\bar{k}_{\text{lowest}}$. That is, like the average approach, the Bayesian approach would lead to the same mean failure stress over all possible triads of tests, and therefore it would lead to the same mean weight of the component.

III. Illustrative Examples

To illustrate the use of Bayesian updating, we will consider a simple case in which both the error and variability are uniformly distributed. We can take $(\sigma_f)_{\text{calc}} = 1.0$ or normalize all results by analytical failure stress. The error bounds are $\pm 10\%$, whereas the variability is bounded by $\pm 15\%$. Three normalized test results are given by $(\sigma_f)_{i,\text{test}}/(\sigma_f)_{\text{calc}} = 1.05$, 1.10, and 1.15 ($i = 1$, 2, 3). Furthermore, we will make a simplifying assumption that the variability of 15% is of the calculated value $(\sigma_f)_{\text{calc}}$, rather of the true value. Then the likelihood function of the first test result becomes

$$f_{1,\text{test}}(\bar{\sigma}_f) = \begin{cases} \frac{1}{0.3(\sigma_f)_{\text{calc}}} & \text{if } \left|\frac{\bar{\sigma}_f - (\sigma_f)_{1,\text{test}}}{(\sigma_f)_{\text{calc}}}\right| \le 0.15 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Because $(\sigma_f)_{\text{calc}} = 1.0$ and the error bounds are $\pm 10\%$, the initial distribution of mean failure stress is $f^{\text{ini}}(\bar{\sigma}_f) = 5$ in the interval (0.9, 1.1) and zero elsewhere (see Fig. 3a). It is then very easy to check from Eq. (7) that the updated distribution with the first test is identical to the initial one, because $f_{1,\text{test}}(\bar{\sigma}_f) = 1/0.3$ is constant in (0.9, 1.2). That is, the first test does not provide any new information (Fig. 3a). For the second test result of 1.1, $f_{1,\text{test}}(\bar{\sigma}_f) = 1/0.3$, but this time only in the interval (0.95, 1.25), so that the updated distribution is zero in (0.9, 0.95) (Fig. 3b). Similarly, the third test of 1.15 excludes the interval (0.95, 1.0) (Fig. 3c), so that finally we have a uniform distribution over (1.0, 1.1). The denominator of Eq. (7) will normalize the area to 1.0, so that we get $f^{\text{upd}}(\bar{\sigma}_f) = 10$ in (1.0, 1.1) and can use the average of 1.05 for our updated mean failure stress $(\bar{\sigma}_f)_{\text{Bayes}}$ (Fig. 3d). Note that this is substantially different from taking the average value of the tests, which is 1.1. Indeed, if we expect that our calculation is correct to within 10%; three tests for which the average is 1.1 should not push us all the way toward this limit.

Next, we have to find the value of explicit knockdown factor. For our uniform variability distribution, the value can be easily calculated to be 0.925 [10]. With the criterion of same average, the knockdown factors are the same for the average and Bayesian approaches: that is, $k_{\text{Bayes}} = k_{\text{avg}}$. This means that if we use the average failure stress of 1.1, the estimated mean failure stress should be $1.1 \times 0.925 = 1.0175$. With the Bayesian updating, the estimated mean failure stress is $1.05 \times 0.925 = 0.9713$.

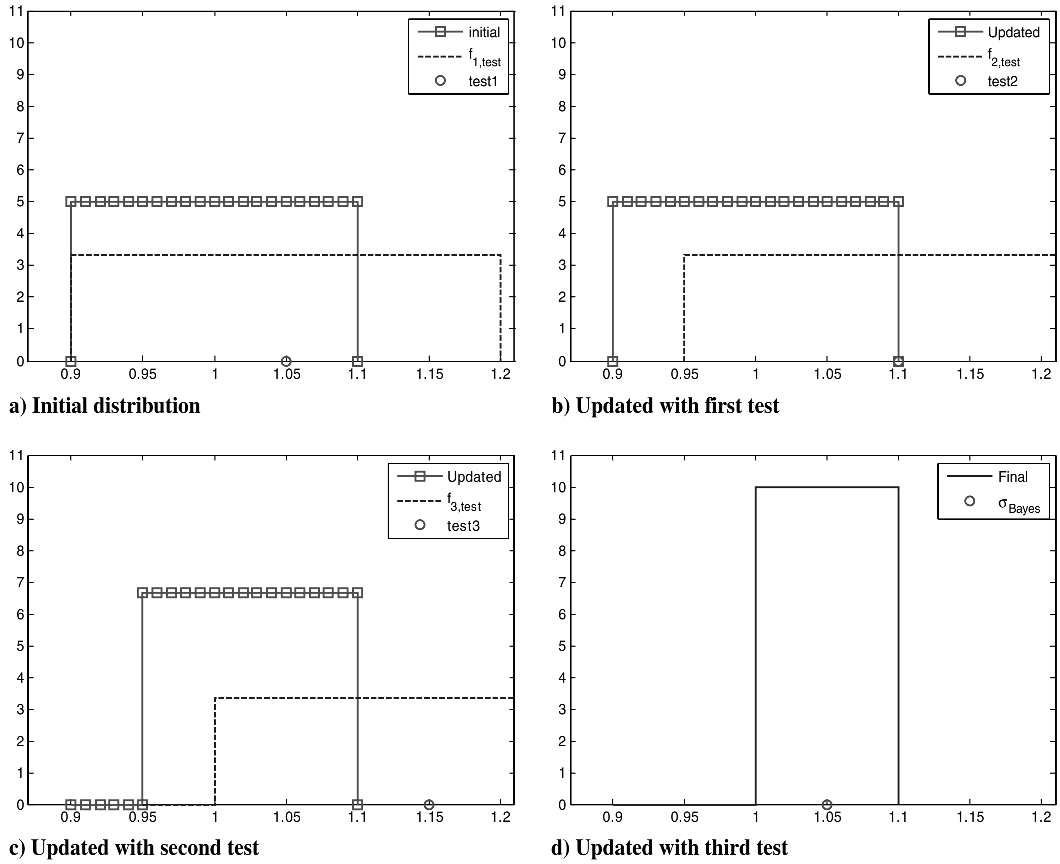
Note that both results are lower than the lowest of our three tests (1.05). This reflects the fact that when the average test result is higher than the calculated value, there is a substantial chance that this is due to variability in material properties, rather than to error. The average approach provides some compensation for this fact, and the Bayesian correction is more drastic because it takes into consideration our confidence in the analytical prediction of the mean failure stress.

For a more typical variability in failure stress, we use a lognormal distribution with a coefficient of variation of 8%. The evolution of distribution of the mean failure stress $f(\bar{\sigma}_f)$ is illustrated in Fig. 4. The average value of the final distribution is 1.0624, and the standard deviation of the final distribution is 0.02826. For the lognormal distribution, the explicit knockdown factor becomes 0.9333 for both average and Bayesian approaches [10]. So we will estimate the mean failure stress to be 1.0266 and with the Bayesian approach to be 0.9915.

In the following, we continue to use the uniform and lognormal distributions as representatives of possible distributions. The uniform distribution corresponds to a situation in which information on variability is very limited and specified by bounds. The lognormal distribution is more typical, though normal and Weibull distributions are also commonly used. Because the two distributions used are very different, using them both illustrates the effect of the distribution on the relative standing of the three approaches.

IV. Variability in Test Results and Its Effect on Conservativeness

To discuss the advantages of the average and Bayesian approaches over the lowest-result approach, we use Monte Carlo simulation



a) Initial distribution

b) Updated with first test

c) Updated with second test

d) Updated with third test

Fig. 3 Illustrative example of Bayesian updating. The initial distribution is uniform in (0.9, 1.1) and the three tests are 1.05, 1.10, and 1.15. The material variability is assumed to be governed by a uniform distribution with variability of 15% of the calculated value. We use the calculated value for simplification. In the rest of the paper, the variability will be governed by $(\bar{\sigma}_f)_{\text{true}}$.

(MCS) to calculate the outcome of the three approaches over a large number of possible true mean failure stresses and possible trials of test results. First, we generate true mean failure stress within our prediction error bounds and three randomly selected samples reflecting possible variability from the mean. We estimate the failure stresses using three approaches and evaluate (by MCS) the c.o.v. of estimated mean failure stress $(\sigma_f)_{\text{est}}$, the chances of unconservative estimates, and the average value of violation in the unconservative estimates. The unconservative estimates are calculated by the ratio of estimates that are higher than our true value, and the average value of violation is the percentile difference between the true mean and our estimation for unconservative cases described already. We consider two cases of material variability in this section: a uniform distribution with $\pm 15\%$ bounds and a lognormal distribution with an 8% c.o.v.

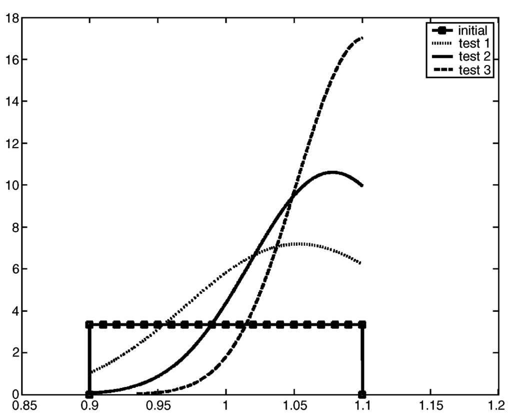


Fig. 4 Evolution of the mean failure-stress distribution $f(\bar{\sigma}_f)$ with Bayesian updating. The initial distribution is uniform in (0.9, 1.1) and the three tests are 1.05, 1.10, and 1.15. The material variability is assumed to be governed by a lognormal distribution with a coefficient of variation of 8%.

With 15% uniformly distributed variability, the standard deviation of the lowest test results estimated by MCS is 0.058 or 6.3% of the mean of the lowest results. However, there is a 12.5% chance that the lowest test data are unconservative estimates of the mean failure stress. On the other hand, the standard deviation of the average test results is 0.05, which is 5% of the mean of the average test result. That is, by using the average test result with an explicit knockdown factor of 0.925, we can achieve the same estimate of mean failure stress on average, but with 20% less volatility. This reduces the chance of unconservative estimate to 5.4%, less than half that of the lowest-result approach. With the Bayesian approach, the standard deviation of the estimated mean failure stress is reduced to 3.2%, and the chance of an unconservative estimate is drastically reduced to 1%.

Not only is the percentage of unconservative estimates reduced, but the mean error in the unconservative estimates is also reduced. For the lowest-result approach the mean over estimate of mean failure stresses for the 12.5% unconservative results is 3.75%. For the 5.4% unconservative results using the mean test result, the average over estimate is only 1.6%, and for the 1% unconservative results in the Bayesian approach, the mean overestimation is only 0.6%.

Figure 5 shows the probability distribution of normalized estimated mean failure stress $(\bar{\sigma}_f)_{\text{est}}/(\bar{\sigma}_f)_{\text{true}}$. Clearly, with the same mean, the average and Bayesian approaches have tempered the variability in test results. Consequently, both approaches have lower chances of unconservative estimated mean failure stress.

Table 1 summarizes the preceding results and provides the corresponding results for the lognormal distribution. We see that the results are similar for the lognormal distribution. For confidence intervals of $\pm 10\%$, the likelihood of unconservative estimate of the mean failure stress is reduced dramatically.

To study the effect of the number of tests, we also considered taking the lowest of five tests instead of three. For the uniform

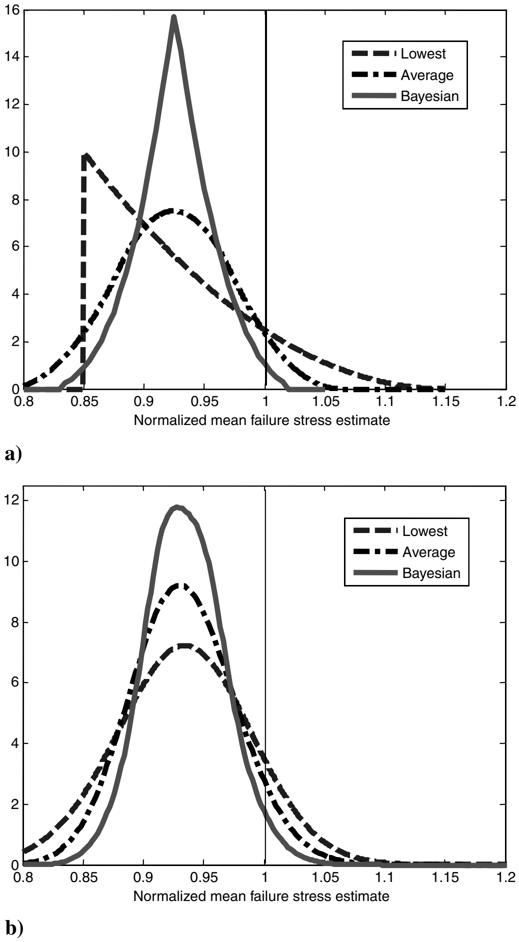


Fig. 5 Distribution of the estimated mean failure stress $(\bar{\sigma}_f)_{est}$ showing how Bayesian updating can reduce unconservative estimates. The material variability is assumed to be governed by a) a uniform distribution of ±15% and b) lognormal distribution with a coefficient of variation of 8%. The area to the right of 1 represents the probability of unconservative estimates.

distribution, this corresponds to a knockdown factor of 0.9, which will guarantee no unconservative results for the Bayesian approach, compared with about 1% with the average approach and 3% with the lowest test result. Significantly, even with five tests, the Bayesian approach still significantly reduces the coefficient of variation in the final estimate to 2.3%, compared with 3.4% for the average approach and 4.7% for the lowest-result approach.

V. Translating Reduced Risk to Reduced Weight

Instead of using the alternative approaches for reducing the chance of unconservative failure-stress estimation, we can trade off some or all of the risk reduction for weight reduction. This can be accomplished by using knockdown factors for the average or Bayesian approaches that are higher than the mean implicit knockdown factor of the lowest-result approach. The higher knockdown factors will lead, on average, to reduced weight of the structural element when the corresponding estimates are used for redesign. Table 2 shows that for the uniform distribution, the knockdown factors can be raised by about 2% for the average approach and 4% for the Bayesian approach. For the lognormal distribution, the knockdown for the average approach may be increased by 1.4 and 2.5%, respectively, using the Bayesian approach. Note that at these values, there is still a small safety advantage in that the mean violation of the unconservative estimates is smaller.

Because the average and Bayesian approaches are based on probabilistic data that may be uncertain, it is possible to settle for smaller weight gains and reduced risk of unconservative failure-stress estimate. Figure 6 shows the variation of the fraction of unconservative results versus the knockdown factor. We will come back to the selection of an intermediate value of the knockdown factor in the next section.

VI. Effect on Uncertainty about Uncertainties

The calculations in Secs. IV and V represent an ideal case in which we have good error bounds and we know the exact variability distribution of test results to apply to Bayesian updating and to set the explicit knockdown factor. In practice, we may tend to select conservative error bounds, because underestimating the error bounds may yield poor results when the error is larger than the bound. We may also have low estimates of variability introduced by the test procedure. In this section, we will therefore consider the following scenarios:

1) In scenario a, the true error bounds are ±10%, but conservative error bounds of ±15% are used in the Bayesian updating.

2) In scenario b, for uniformly distributed variability, Bayesian updating uses 15% variance, but the actual variance is 20%. For the lognormal case, Bayesian updating uses an 8% c.o.v., but the actual c.o.v. is 10%. Note that the latter increase would correspond to an additional 6% variability from an independent source such as testing variability on top of the 8% material variability. However, the calculation of explicit knockdown factors is based on the material variability alone, and the values are 0.925 and 0.933, respectively. Similarly, the Bayesian updating is performed with material variability alone.

3) In scenario c, there is a combination of both error-bound increases and underestimates of variance.

Scenario a is discussed in Table 3. The table shows that being conservative and overestimating the error by 50% reduces the advantage of the Bayesian approach, but still leaves a substantial margin of improvement compared with the results in Table 1. Compared with the lowest-result approach, the probability of unconservative estimate of the mean failure stress is still reduced from 12.5 to 1.8% for uniform variability and from 11.3 to 4.3% for the lognormal variability case.

The results corresponding to scenarios b and c are summarized in Table 4. Because the actual variability is higher than the assumed one, the mean of the implicit knockdown factors $\bar{k}_{lowest}$ is lower than the calculated one. For the uniform distribution, the explicit knockdown factor is 0.925, compared with the implicit 0.9 for the lowest-result approach, which means that we will also be able to use stress allowables with the Bayesian approach that are higher by about 2.7%. This is likely to correspond to a similar reduction in weight.

Table 1 Comparison of c.o.v. and likelihood of unconservative estimates using the lowest, average, and Bayesian approaches[a]

Variability	Uniform (±15% bounds)			Lognormal (8% c.o.v.)		
	Lowest	Average	Bayesian	Lowest	Average	Bayesian
% c.o.v.	6.3	5.0	3.2	5.9	4.6	3.3
Knockdown factor[b]	0.925	0.925	0.925	0.9333	0.9333	0.9333
Unconservative, %	12.5	5.4	1.0	11.3	6.4	2.8
Mean unconservative, %	3.75	1.6	0.6	2.7	2.1	1.3

[a]The Bayesian approach is based on 10% confidence in the calculated result with uniform distribution in (0.9, 1.1). Results are based on 500,000 MCS.
[b]Explicit for the average and Bayesian approaches and mean of the implicit knockdown factor for the lowest-result approach.

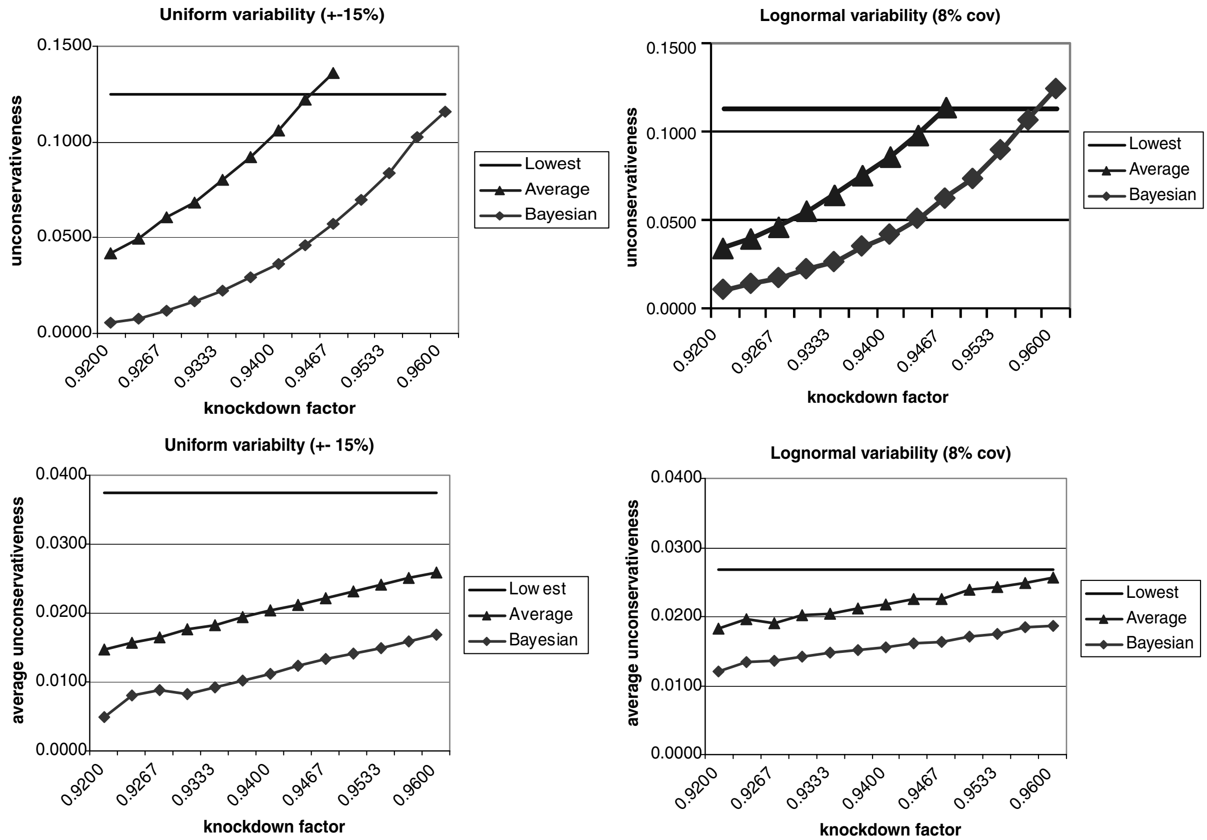


Fig. 6 Effect of knockdown factor k_f. Choosing a knockdown factor k_{avg} or k_{Bayes} that is higher than $\bar{k}_{lowest}$ will eventually lead to cost savings by reducing weight. Furthermore, with the same degree of unconservative result, the mean violation in unconservative estimates still shows some advantages in safety measure. This figure is based on 200,000 MCS result at each point. The waviness of the curves is due to the variability of MCS.

For the lognormal distribution, the corresponding values are 0.9333 and 0.9172, or a 1.7% gain.

For scenario b, in which only the variability is underestimated, some test results may expose this underestimate. For example, 10% error plus the actual variability of a uniform 20% may lead to a test result in which the failure stress is as high as $1.10 \times 1.20 = 1.320$. This is beyond our predicted limit for the Bayesian approach and will give a numerical error in the integration procedure. For generating the results in Table 4, we assume that this will lead us to select highest limit of error (10%). So the estimated mean failure stress will be $1.1 \times 0.925 = 1.0175$ for the extreme cases in uniform variability.

With the Bayesian approach applied for this scenario, we achieve the higher stress allowable with a reduction of more than a factor of 3 in the likelihood of having unconservative mean failure-stress estimates (3.4% compared with 12.5%) for the uniform distribution and better than a factor of 2 (4.7% compared with 11.1%) for the lognormal distribution. Furthermore, these gains in stress allowables and reductions in likelihood of unconservative estimates of mean

Table 2 Comparison of three approaches when the knockdown factor is chosen such that all approaches have the same likelihood of unconservative estimate[a]

Variability	Uniform (±15% bounds)			Lognormal (8% c.o.v.)		
	Lowest	Average	Bayesian	Lowest	Average	Bayesian
Knockdown factor[b]	0.925	0.9443	0.9615	0.9333	0.9467	0.9580
Unconservative, %	12.5	12.5	12.5	11.3	11.3	11.3
Mean unconservative, %	3.75	2.1	1.7	2.7	2.3	1.8

[a]Even with the same degree of unconservativeness, the mean violation amounts of unconservative estimates are smaller than the lowest-result approach. Results are based on 500,000 MCS.
[b]Explicit for the average and Bayesian approaches and mean of the implicit knockdown factor for the lowest-result approach.

Table 3 Comparison of c.o.v. and likelihood of unconservative estimates using 10 and 15% confidence interval in the Bayesian approach with scenario a

Variability	Uniform (±15% bounds)			Lognormal (8% c.o.v.)		
	Lowest	Bayesian ±10%	Bayesian ±15%	Lowest	Bayesian ±10%	Bayesian ±15%
% c.o.v.	6.3	3.2	3.5	5.9	3.3	3.7
Knockdown factor[b]	0.925	0.925	0.925	0.9333	0.9333	0.9333
Unconservative, %	12.5	1.0	1.8	11.3	2.8	4.3
Mean unconservative, %	3.75	0.6	1.0	2.7	1.3	1.7

[a]The Bayesian approach based on 10% confidence interval is already given in Table 1. We compared it with the Bayesian approach based on 15% confidence in the calculated result with uniform distribution in (0.85, 1.15), whereas the true value is in (0.9, 1.1). Results are based on 500,000 MCS.
[b]Explicit for the average and Bayesian approaches and mean of the implicit knockdown factor for the lowest-result approach.

Table 4 Comparison of c.o.v. and likelihood of unconservative estimates using the lowest-case, average, and Bayesian approaches[a] for scenario b when true variability is higher than the variability used for the Bayesian approach (scenario c is shown in parentheses)

Variability	Uniform (±20% bounds)			Lognormal (10% c.o.v.)		
	Lowest	Average	Bayesian	Lowest	Average	Bayesian
% c.o.v.	8.6	6.66	4.0 (4.7)	7.4	5.8	3.8 (4.5)
Knockdown factor[b]	0.9	0.925	0.925	0.9172	0.9333	0.9333
Unconservative, %	12.5	11.8	3.4 (5.8)	11.1	11.0	4.7 (7.1)
Mean unconservative, %	5.0	2.75	1.2 (1.7)	3.3	2.8	1.9 (2.2)

[a]The Bayesian approach is based on a 10% confidence in the calculated result with uniform distribution in (0.9, 1.1). The numbers in parentheses are for 15% confidence intervals. Bayesian approach assumes 15% uniform or 8% lognormal variability. Results are based on 500,000 MCS.
[b]Explicit for the average and Bayesian approaches and mean of the implicit knockdown factor for the lowest-result approach.

failure stresses are accompanied by substantial reductions in the magnitude of the mean unconservative error: from 5 to 1.2% for the uniform distribution and from 3.3 to 1.9% for the lognormal distribution.

For scenario c, which combines overestimate of the error with underestimate of the variability, gains are diminished but are still substantial. For the uniform distribution, gains in stress allowables are still accompanied by better than a factor-of-2 reduction in the likelihood of unconservative estimates (5.8% compared with 12.5%) for the uniform distribution and almost a factor-of-2 reduction for the lognormal distribution (7.1% compared with 11.1%). Again, these gains are in addition to about a 2% increase in the knockdown factor, corresponding to significant weight savings.

VII. Conclusions

Test results of structural elements are often used in a conservative fashion by updating estimates of mean failure stress based on the lowest result of the tests. This is equivalent to applying a highly variable implicit knockdown factor. We showed that it is more effective to use less-variable estimates of the mean failure stress and apply an explicit knockdown factor. We compared two alternative approaches with the lowest-result approach: one based on average test results and the other based on a Bayesian update of an assumed error distribution.

Examples with uniform and lognormal distributions of test results were used to compare the lowest-result approach with the two alternatives with explicit knockdown factors. Both approaches were shown to yield large reductions in the likelihood of unconservative estimates of the mean failure stresses. The average approach reduced this likelihood by about a factor of 2 and the Bayesian approach by up to an order of magnitude (from 12.5 to 1%). In addition, the magnitude of the mean lack of conservativeness was also greatly reduced. It is possible to replace these safety gains by weight reductions by using higher knockdown factors for the average and Bayesian approaches. For the examples considered here, it is possible to obtain 1.4–2% weight savings using the average approach and 2.7–4% savings using the Bayesian approach.

We next examined scenarios in which the estimates of error and variability are substantially inaccurate. Specifically, we considered a scenario in which error estimates are inflated from 10 to 15%. Then we considered a scenario in which experimental errors are almost as high as material variability, increasing the combined coefficient of variation from 8 to 10%. Finally, we considered a scenario combining both inflated error estimates and underestimated variability. We showed that even under these adverse conditions, there are still substantial gains in reduced likelihood of unconservative estimates of mean failure stresses. Remarkably, the underestimate of variability also results in about a 2% higher average of estimated mean failure stresses. Thus, we are able to simultaneously use higher mean failure stress and reduce the likelihood of unconservative estimates.

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