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RESEARCH PAPER

A critical evaluation of asymptotic sampling method for highly safe structures

Gamze Bayrak[1] · Erdem Acar[1]

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Abstract
Asymptotic sampling is an efficient simulation-based technique for estimating small failure probabilities of structures. The concept of asymptotic sampling utilizes the asymptotic behavior of the reliability index with respect to the standard deviations of the random variables. In this method, the standard deviations of the random variables are progressively inflated using a scale parameter to obtain a set of scaled reliability indices. The collection of the standard deviation scale parameters and corresponding scaled reliability indices are called support points. Then, least square regression is performed using these support points to establish a relationship between the scale parameter and scaled reliability indices. Finally, an extrapolation is performed to estimate the actual reliability index. The accuracy and performance of the asymptotic sampling method are affected by various factors including the sampling method used, the values of the scale parameters, the number of support points, and the formulation of extrapolation models. The purpose of this study is to make a critical evaluation of the performance of the asymptotic sampling method for highly safe structures, and to provide some guidelines to improve the performance of asymptotic sampling method. A comprehensive numerical procedure is developed, and structural mechanics example problems with varying number of random variables and probability distribution types are used in assessment of the performance of asymptotic sampling method. It is found that generating the random variables by Sobol sequences and using the 6-model mean extrapolation formulation give slightly more accurate results. Besides, the optimum initial scale parameter is approximately around 0.3 and 0.4, and the optimum number of support points is typically four for all problems. As the reliability level increases, the optimum initial scale parameter value decreases, and the optimum number of support points increases.

Keywords Asymptotic behavior · Extrapolation models · Reliability index · High reliability

1 Introduction

Structural reliability is predicted using a limit state function (or performance function) that is used to separate the safe and failure regions of an input space. The probability of failure estimation requires calculation of the multi-dimensional integral of the joint probability density function of all the random variables over the failure region

$$P_f = \int \cdots \int I\left[g(\mathbf{x}) \leq 0\right] f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where I is the indicator function that takes the value of 1 when the condition is true and 0 when the condition is false, $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint probability density function of the set of random variables $\mathbf{X}$, and $g(\mathbf{x})$ is the limit state function. For most real life structural problems the analytical integration of this multi-dimensional function is not possible; therefore, analytical and simulation-based approaches have been proposed to estimate failure probability.

Analytical approaches require a small number of limit state function calculations; therefore, they are typically computationally inexpensive compared with simulation-based approaches. The most popular analytical methods are the first-order (Hasofer and Lind 1974; Rackwitz and Fiessler 1978) and second-order reliability methods (Breitung 1984;

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Special Issue dedicated to Dr. Raphael T. Haftka

✉ Erdem Acar
acar@etu.edu.tr

[1] Department of Mechanical Engineering, TOBB University of Economics and Technology, Söğütözü, 06560 Ankara, Turkey

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Tvedt 1990), which are based on the first-order and second-order expansions of the limit state function at the most probable failure point (MPP), respectively. Although the analytical approaches are computationally advantageous compared to other methods, they are not necessarily suitable for real-life problems which have complex and nonlinear limit state functions (e.g., problems with multiple failure modes).

Simulation-based approaches can yield accurate results provided that a sufficient number of simulations are applied. The most popular simulation-based approach is the Monte Carlo simulation (MCS) method (Rubinstein and Kroese 2016). Unfortunately, MCS is computationally expensive for estimating small failure probabilities. Variance reduction techniques such as importance sampling (Melchers 1989) and adaptive importance sampling (Wu 1994) can be used to improve the accuracy of failure probability estimations. These methods rely on the concept of MPP search and most MPP search algorithms may fail or yield erroneous results when the limit state function is highly nonlinear or discontinuous. In such cases, simulation-based methods that do not rely on MPP search such as stratified sampling (Iman and Conover 1980), subset simulation (Au and Beck 2001) or line sampling (Koutsourelakis et al. 2004) can be used.

Other alternatives include the utilization of metamodels such as Kriging (Kaymaz 2005; Xiao et al. 2020; Zhou et al. 2020), neural networks (Gondal and Lee 2012; Papadopoulos et al. 2012), support vector regression (Basudhar and Missoum 2010), radial basis functions (Zhou et al. 2019a, b), and polynomial chaos expansions (Diaz et al. 2018; Zhou et al. 2019a, b). Owing to the extremely large computational cost of implicit functions in certain problems, surrogate models are widely used as a replacement (Jiang et al. 2019; Chojaczyk et al. 2015). However, the performances of surrogate models are affected by the curse of dimensionality when the dimensions are large. That is, the computational effort required to construct a surrogate model grows dramatically with the number of random input variables.

Extrapolation-based methods are used to overcome the disadvantages of other methods used in high reliability systems. Asymptotic sampling (Bucher 2009) is an extrapolation-based method for estimating small failure probabilities. This method extrapolates from low reliability indices to the high reliability indices based on the asymptotic behavior of the failure probability with respect to the standard deviation of the variables. By using a scale parameter, the standard deviations of the random variables are progressively inflated to obtain various (smaller) scaled reliability indices that can be predicted accurately using a small number of samples. Subsequently, least squares regression is used to establish a relationship between the standard deviation inflation parameter and scaled reliability index values. Finally, extrapolation is performed to estimate the actual reliability index. This method can reduce the computational cost for the estimation of the high reliability index due to the fact that the low reliability index can be estimated with lower computation. Zhangchun et al. (2013, 2014) improved the accuracy of the asymptotic sampling method using mean prediction of various extrapolation models. Acar (2016) increased the effectiveness of the asymptotic sampling by re-formulating the extrapolation formulation for highly safe structures with separable limit state functions.

The aim of this paper is to critically evaluate the performance of the asymptotic sampling method, and to provide guidelines to improve it. A comprehensive numerical procedure is developed, and example problems with varying numbers of random variables and probability distribution types are used to assess the performance of asymptotic sampling method.

Prof. Raphael (Rafi) T. Haftka developed novel approaches in the field of reliability prediction. The examples include the probabilistic sufficiency factor (Qu and Haftka 2004), multiple tail median (Ramu et al. 2010), conservative reliability estimation with bootstrapping (Picheny et al. 2010), and separable Monte Carlo (Smarslok et al. 2010; Chaudhuri and Haftka 2013), to name a few. In particular, the multiple tail median formulation (Ramu et al. 2010) inspired Zhangchun et al. (2013, 2014) to develop a mean extrapolation formulation, which will be covered in Sect. 3. This paper is dedicated to the memory of Prof. Rafi Haftka.

The remainder of this paper is organized as follows: The asymptotic sampling method is described briefly in the next section. The mean extrapolation formulation used in asymptotic sampling is presented in Sect. 3. The numerical examples and computational procedure used in this study are discussed in Sect. 4. The results obtained from these example problems are presented and discussed in Sect. 5. Finally, a summary of the important conclusions is given in Sect. 6.

2 Asymptotic sampling

Bucher (2009) developed an asymptotic sampling method that enables accurate estimation of high reliability indices. In this method, the standard deviation of the random variables are artificially inflated using a scale parameter to obtain smaller reliability indices, known as "scaled" reliability indices. Subsequently, a functional relationship is established between the scale parameters and scaled reliability indices. Finally, the actual reliability index is predicted using the established functional relationship.

Bucher first considered a problem involving a linear limit state function and suggested that this problem can be reduced to a single variable with standard deviation of σ via an appropriate coordinate transformation. Then, the reliability index can be formulated as

$$\beta(f) = \frac{\beta_f}{f} \tag{2}$$

where f is the scale factor and β_f is the scaled reliability index computed for the scaled standard deviation of the random variable $\sigma_f = \sigma/f$. The actual reliability index can be computed using $\beta_{\text{act}} = \beta(f = 1)$. It is noteworthy that for problems with multiple input random variables, the standard deviations of all random variables are scaled using the same scale factor f.

To obtain a good estimate for β_{act}, the reliability index for a larger value of σ (a smaller value of the scale factor f) can be computed using MCS, and then simply extrapolated by multiplying the result with f.

Additionally, Bucher considered a second analytical problem with a (hyper)circular limit state function in n-dimensional Gaussian space in which failure is expressed as $g(X) = R^2 - X^T X \leq 0$. In this case, the reliability index is expressed in terms of the χ^2-distribution with n degrees of freedom, as follows:

$$\beta = \Phi^{-1}[1 - \chi^2(f^2R^2, n)] \tag{3}$$

This relationship between reliability index and the standard deviation scale parameter f is shown in Fig. 1.

Based on the asymptotic behavior of the reliability index with respect to the standard deviation scale parameter, Bucher assumed the following functional relationship between the reliability index and the standard deviation scale parameter f

$$\beta = Af + \frac{B}{f} \tag{4}$$

Notice that as $f \to \infty$ (that is, as $\sigma_f \to 0$) the reliability index $\beta \to \infty$ so that the asymptotic behavior is ensured. Coefficients A and B are determined from least squares regression analysis based on the estimates of β for different values of f smaller than 1. That is, a set of "support points" $[f_i, \beta(f_i)]$ shown in Fig. 2 is used in the regression.

To assign equal weights to all support points for the regression analysis, Eq. (4) can be rewritten in terms of a scaled reliability index as follows:

$$\frac{\beta}{f} = A + \frac{B}{f^2} \tag{5}$$

For this method, it is essential to use a sampling method that yields stable results. A typical choice is Latin hypercube sampling (LHS) method (Iman 1982; Florian 1992). Alternatively, pseudo-random sequences with low-discrepancy sampling methods such as Sobol sequences (Bratley and Fox 1988), Halton sequences (Halton 1960) or Good lattice point sets (Fang and Wang 1994) can be utilized. In this study, the performance of the asymptotic sampling method is investigated by sampling with the commonly used LHS and Sobol sequences.

Bucher (2009) initiated the asymptotic sampling algorithm using the scale parameter, $f_0 = 1$. The required number of samples in the failure domain was set to $N_0 = 10$. In the first step, the actual number of samples N_F in the failure domain was inadequate (less than N_0). Therefore, the parameter f was decreased by a factor of 0.9, and the simulation was repeated until N_F was equal to or exceeded N_0. The support points and regression curve obtained from the extrapolation process are shown in Fig. 2.

Bucher (2009) stated that five support points can be used. In his later studies, he used different numbers of support points. In a follow-up study, Gasser and Bucher (2018) suggested that four or more support points to yield a more stable regression; however, this practice resulted in an increase in computational effort. They used different numbers of support points in different example problems without a clear statement on determination of the proper number.

The extrapolation performance is affected by the number of support points (N_s), initial scale parameter (f_0) and

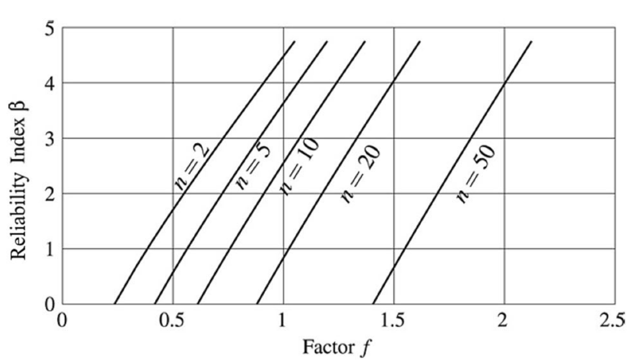

Fig. 1 Relationship between the reliability index and the standard deviation scale parameter f for hyper circular limit state function (Bucher 2009)

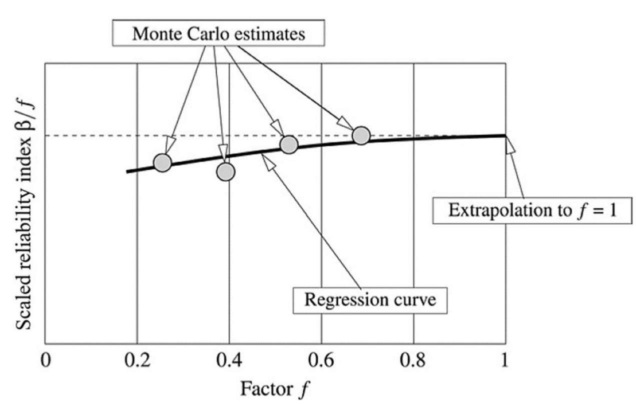

Fig. 2 The concept of asymptotic sampling (Bucher 2009)

sampling methods used to generate the random variables. In this study, we aim to provide guidelines for these parameters to improve the asymptotic sampling method.

3 Mean extrapolation technique

As noted earlier, the asymptotic sampling method extrapolates the high reliability index from the low reliability indices obtained. This technique can decrease the computational cost for the evaluation of a high reliability index because the low reliability index can be estimated at a lower computational cost. However, Zhangchun et al. (2013) discovered that the use of a single extrapolation model was not robust. Inspired by Prof. Rafi Haftka's multiple tail median formulation (Ramu et al. (2010)), where the median of multiple tail model predictions is used, Zhangchun et al. (2013) proposed to generate multiple extrapolation models and use the mean value of the reliability predictions of these models. Specifically, they proposed using 10-extrapolation models, as expressed in Eqs. (6) and (7).

$$\beta_t(f) = A_t f + \frac{B_t}{f^{q_t}}\left(t = 1, 2, 3, 4, 5; q_1 = 3, q_2 = 2, q_3 = 1, q_4 = 0.5, q_5 = \frac{1}{3}\right) \quad (6)$$

$$\beta_t(f) = A_t f + \frac{B_t}{exp(f^{q_t})}\left(t = 6, 7, 8, 9, 10; q_6 = 3, q_7 = 2, q_8 = 1, q_9 = 0.5, q_{10} = \frac{1}{3}\right) \quad (7)$$

where $t=1,\ldots,10$ represents the extrapolation model index, q_t ($t=1, \ldots, 10$) is the exponent of the extrapolation model, and exp(.) is the exponential operation with natural base e as the base. The coefficients A_t and B_t are determined through least squares regression. Then, the actual reliability index is computed using the average of these 10 extrapolation models, expressed as:

$$\beta(1) = \frac{1}{10}\sum_{t=1}^{10}\beta_t(1) = \frac{1}{10}\left(\sum_{t=1}^{5}\left(A_t + B_t\right) + \sum_{t=6}^{10}\left(A_t + B_t/e\right)\right) \quad (8)$$

In a follow-up study, Zhangchun et al. (2014) proposed a new mean extrapolation technique that involves 6 extrapolation models to estimate the actual reliability index. In that study, only models corresponding to q_2, q_3 and q_4 in Eq. (6), and the models corresponding to q_7, q_8 and q_9 in Eq. (7) are used. Zhangchun et al. (2014) did not provide a comparison of the accuracies of these two versions; hence, in this study, we aim to provide this comparison.

As noted earlier, the studies of Zhangchun et al. (2013, 2014) were inspired by the study of Ramu et al. (2010), where the median of multiple predictive models are used. Therefore, in this study, the accuracies of the mean and median extrapolation models are also compared.

4 Numerical examples and procedure

This section provides six structural mechanics example problems to investigate the effects of various parameters (including the sampling method, values of scale parameters, number of support points, and extrapolation model formulations) on the performance of the asymptotic sampling method. To reduce random sampling effect, all asymptotic sampling processes are repeated 1000 times. The average numbers of limit state function evaluations (*NFE*) are stored such that the computational costs can be analyzed for each problem. The performance of the asymptotic sampling is measured through normalized RMSE ($RMSE_{\mathrm{nor}}$) values obtained from 1000 runs, where $RMSE_{\mathrm{nor}}$ is computed from:

$$RMSE_{\mathrm{nor}} = \frac{RMSE}{\beta_{\mathrm{act}}} \quad (9)$$

To explore the effect of the sampling method, LHS and Sobol sequences are used for all example problems and all reliability levels, and $RMSE_{\mathrm{nor}}$ values obtained are compared. In this study, the number of samples is limited to 512, based on Bucher (2009). To investigate the effect of the initial scale factor, we compare the $RMSE_{\mathrm{nor}}$ values corresponding to different initial scale parameter (f_0) values ranging from 0.2 to 1.0. To explore the effect of the number of support points, we compute $RMSE_{\mathrm{nor}}$ values for the number of support points of 2, 3, 4 and 5, and compare them to find the proper number of support points. In this study, we use 10% $RMSE_{\mathrm{nor}}$ value as threshold value while finding the proper values of the abovementioned parameters with smallest number of limit state function evaluations. We investigated the performance of the asymptotic sampling for these example problems for the reliability index values of 4, 4.5, 5, 5.5, and 6, which correspond to the failure probabilities of 3.17×10^{-5}, 3.40×10^{-6}, 2.87×10^{-7}, 1.90×10^{-8}, and 9.87×10^{-10}, respectively.

All the example problems presented herein are selected as structural mechanics problems. The first example problem is a simple two-variable problem involving a linear limit state function; therefore, the analytical solution can easily be obtained. Starting from this simple example, the dimensionality and the nonlinearity of the functions are varied. Here, the dimensionality refers to the number of variables, and the nonlinearity is measured with the square root of the mean of the square of the deviation of the response from the linear response surface, inspired from Emancipator and Kroll (1993). For each problem, a linear response surface is constructed in the range of mean plus/minus six times the standard deviation of the input random variables. Finally, our nonlinearity metric (NM) is defined as the root mean square deviation (defined above) normalized with the range of the corresponding response (see Eq. 10). The linear response surfaces are constructed by generating 10,000 data points in the input random variable range mentioned above. Table 1 shows the dimensionality and nonlinearity of the example problems used in the paper.

$$NM = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{y}_i - y_i\right)^2}}{y_{\max} - y_{\min}} \times 100 \tag{10}$$

where n = 10,000, $\hat{y}_i$ is the value of the response predicted by the linear response surface, y_i is the actual value of the response, $y_{\max}$ and $y_{\min}$ are the maximum and minimum values of the response used in construction of the linear response surface.

4.1 Connecting rod problem

The connecting rod problem under axial loading is illustrated in Fig. 3. The problem is a two-variable simple problem involving the following linear limit state function

$$g = C - R \tag{11}$$

where R and C denote stress and strength, respectively, and both are random variables. The statistical properties of the random variables are listed in Table 2.

The mean value of the stress μ_R can be changed to obtain various levels of reliability index values. For this problem, the actual reliability index can be obtained easily using Eq. (12) as the limit state function is linear and both random variables follow normal distribution. In Eq. (12), μ and σ correspond to the mean and standard deviation of the corresponding quantity, respectively.

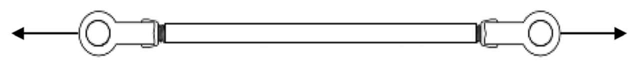

Fig. 3 The connecting rod under axial loading

$$\beta = \frac{\mu_C - \mu_R}{\sqrt{\sigma_C^2 + \sigma_R^2}} = \frac{100 - \mu_R}{\sqrt{8^2 + 6^2}} = 10 - \frac{\mu_R}{10} \tag{12}$$

4.2 Cantilever beam problem

The cantilever beam problem (Wu et al. 2001) is illustrated in Fig. 4. The limit state occurs when the tip displacement exceeds the allowable, D_o.

$$g = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \tag{13}$$

where E is the modulus of elasticity, X and Y are mutually independent random loads, and width $w = 2.7''$ and thickness $t = 3.4''$ are the design parameters. The definitions of the random variables are presented in Table 3. The allowable displacement D_0 can be varied to obtain various reliability levels as given in "Appendix A".

4.3 Central crack problem

In this example (Bayrak and Acar 2018), a rectangular plate of finite width W with a central through-thickness crack of length $2a$ loaded in tension with a uniform stress, S, is considered (see Fig. 5). The limit state function for this problem can be written as:

$$g = K_{IC} - \sqrt{\sec\left(\frac{\pi a}{W}\right)}S\sqrt{\pi a} \tag{14}$$

where a is the half crack length, W is the plate width, S is the applied stress, K_{IC} is the fracture toughness, and all these variables are taken random. The probability distributions as well as the mean and the standard deviations of the random variables are given in Table 4. The mean value of the

Table 1 The dimensionality and nonlinearity of the example problems

ID	Problem	Dimensionality (n_{var})	Nonlinearity metric (NM)
1	Connection rod	2	0
2	Cantilever beam	3	4.31
3	Central crack	4	0.48
4	Fortini's clutch	4	6.02
5	Roof truss	6	4.25
6	I beam	8	2.40

Table 2 Statistical properties of the random variables in the connecting rod problem

Random variable	Distribution	Mean	Standard deviation
R	Normal	μ_R	6
C	Normal	100	8

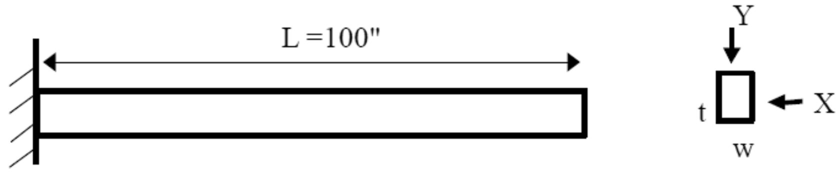

Fig. 4 The cantilever beam under vertical and lateral bending

Table 3 Statistical properties of the random variables in the cantilever beam problem

Random variable	Distribution	Mean	Standard deviation
X (lb)	Normal	500	100
Y (lb)	Normal	1000	100
E (psi)	Normal	29×10^6	1.45×10^6

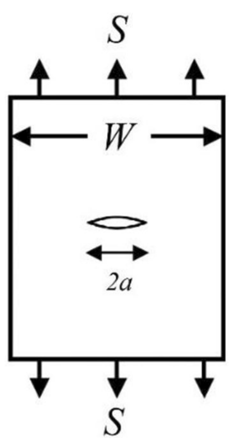

Fig. 5 Central cracked plate with a finite width

Table 4 Statistical properties of the random variables in the central crack problem

Random variable	Distribution	Mean	Standard deviation
a (mm)	Normal	25	0.75
W (mm)	Normal	500	5
S (MPa)	Normal	100	10
K_{IC} $(MPa\sqrt{m})$	Normal	$\overline{K}_{IC}$	0.1 $\overline{K}_{IC}$

fracture toughness $\left(\overline{K}_{IC}\right)$ is varied to adjust the reliability level (see "Appendix A").

4.4 Fortini's clutch problem

The Fortini's clutch, used in many tolerance analysis literature (Creveling 1997), is illustrated in Fig. 6. The contact angle y is given in terms of the independent component variables, X_1, X_2, X_3, and X_4 as follows:

$$y = \arccos\left(\frac{X_1 + 0.5(X_2 + X_3)}{X_4 - 0.5(X_2 + X_3)}\right) \tag{15}$$

The statistical properties of the random variables are presented in Table 5. The limit state function of this problem is expressed as follows

$$g = y - y_{\text{crit}} \tag{16}$$

where y_{crit} can be customized to obtain various reliability levels, as presented in "Appendix A".

4.5 Roof truss problem

A roof truss subject to uniform loads, introduced by Song et al. (2009), is shown in Fig. 7. The top boom and compression members are concrete, and the bottom boom is made of steel. The limit state function is expressed as follows:

$$g = c - \left(\frac{ql^2}{2}\right)\left(\frac{3.81}{AcEc} + \frac{1.13}{AsEs}\right) \tag{17}$$

where c is the vertical deflection at the peak of the structure (node C in Fig. 7), q is uniform load, l is length, A_s and A_c are sectional areas and E_s and E_c are the modulus of elasticity. The definitions of the random variables are presented in Table 6. The value of the vertical deflection c can be changed to arrange the reliability level of the problem as given in "Appendix A".

4.6 I-beam problem

A simply supported I-beam illustrated in Fig. 8. The beam is subjected to a concentrated load as discussed in Huang and Du (2006). This problem involves a limit state function, which is defined as the difference between the strength (S) and the maximum normal stress ($\sigma_{\max}$) due to bending, expressed as follows:

$$g = S - \sigma_{\max} \tag{18}$$

where

$$\sigma_{\max} = \frac{Pa(L-a)d}{2LI}; I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12} \tag{19}$$

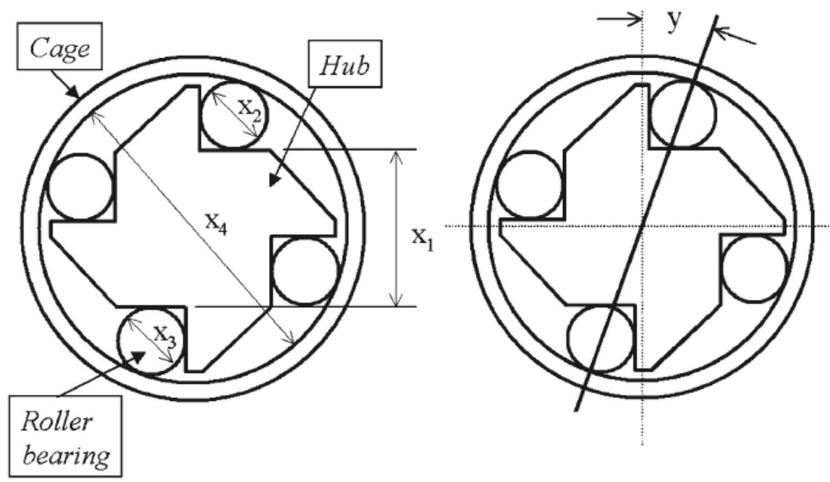

Fig. 6 Fortini's clutch (Lee and Kwak 2006)

Table 5 Statistical properties of the random variables in the Fortini's clutch problem

Random variable	Distribution	Mean	Standard deviation
X_1 (mm)	Lognormal	55.29	0.0793
X_2 (mm)	Normal	22.86	0.0043
X_3 (mm)	Normal	22.86	0.0043
X_4 (mm)	Extreme type I	101.6	0.0793

For X_1 the scale parameter is $\lambda = 4.01$, and shape parameter is $\zeta = 0.0014$. For X_4, the location parameter is $\mu = 101.6$, and scale parameter is $\beta = 0.062$

Table 6 Statistical properties of the random variables in the roof truss problem

Random variable	Distribution	Mean	Standard deviation
q (kN)	Normal	20×10^3	1400
l (m)	Normal	12	0.12
A_s (m^2)	Normal	9.82×10^{-4}	5.892×10^{-5}
A_c (m^2)	Normal	0.04	4.8×10^{-3}
E_s (GPa)	Normal	1×10^{11}	6×10^9
E_c (GPa)	Normal	2×10^{10}	1.2×10^9

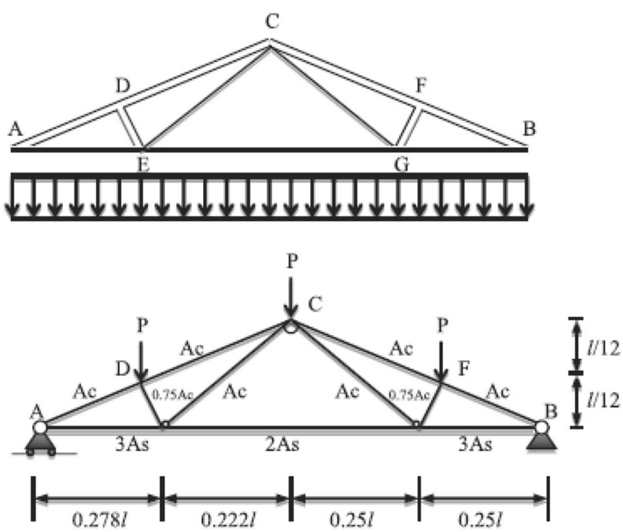

Fig. 7 Roof truss

The statistical properties of the random variables in this example are listed in Table 7. The mean value of the strength can be tailored to accommodate the reliability level of the problem, as given in "Appendix A".

5 Results

5.1 Effect of sampling method

Bucher (2009) suggests using low-discrepancy sampling methods (e.g., Sobol sequences, Halton Sequences, etc.) to populate a variable space more uniformly such that more stable support points can be obtained. In this study, we consider two well-known sampling methods, namely LHS and Sobol sequences, and compare their effectiveness in asymptotic sampling. The results obtained using these two sampling methods are compared based on both the 6-model and 10-model mean extrapolation formulations.

While comparing the results, we examine the average of the RMSE values ($RMSE_{avg}$) obtained for all problems at each reliability level. Although no significant difference is observed between them, the use of the Sobol sequences yields smaller estimation errors, as expressed in terms of RMSE of the reliability indices, particularly at higher reliability levels (see Fig. 9). Hence, we conclude that using Sobol sequence is more efficient than using LHS; therefore

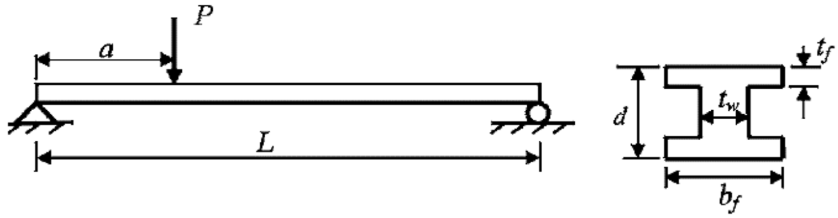

Fig. 8 The cross section and loading for the simply supported I beam

Table 7 Statistical properties of the random variables in the I beam problem

Random variable	Distribution	Mean	Standard deviation
P	Normal	6070	200
L	Normal	120	6
a	Normal	72	6
S	Normal	$\overline{S}$	0.15 $\overline{S}$
d	Normal	2.3	1/24
b_f	Normal	2.3	1/24
t_w	Normal	0.16	1/48
t_f	Normal	0.26	1/48

we carry out sampling with Sobol sequences in the subsequent studies. The abovementioned finding is consistent with the suggestion of Bucher (2009), who reported that low-discrepancy sampling methods populate the variable space more uniformly, thereby affording more stable support points. As the support points become more stable, the RMSE values decrease. It is noteworthy that the RMSE values presented in Fig. 9 are 0.55 in terms of reliability index; however, it might result in larger error values in terms of the probability of failure.

5.2 Effect of extrapolation model formulations

In this section, a comparison of the 6-model and 10-model extrapolation formulations proposed by Zhangchun et al. (2003, 2004) is presented. We compare these formulations in terms of two aspects: the performance of (i) the best individual extrapolation model, (ii) the mean extrapolation model. In this paper, we call the models in Eq. (6) as "*nor* q_3", "*nor* q_2", "*nor* q_1", "*nor* $q_{0.5}$" and "*nor* $q_{1/3}$", in the order they appeared in the equation. Similarly, we call the models in Eq. (7) as "*exp* q_3", "*exp* q_2", "*exp* q_1", "*exp* $q_{0.5}$" and "*exp* $q_{1/3}$", in the order they appeared in the equation.

Considering the performance of the best individual models in both extrapolation formulations, it is discovered that the RMSE of the best individual model in the 10-model extrapolation formulation is always smaller than or equal to that of the 6-model extrapolation formulation (see Fig. 10a), as expected. Table 8 shows that the "*nor* q_2" and "*nor* q_3" models are the most accurate models of the 6-model and 10-model extrapolation formulations for most of the reliability levels, respectively. For the roof truss problem, "*nor* q_1" is found to be the most accurate model for all reliability levels except $\beta = 6$, and "*nor* $q_{0.5}$" is found to be the most accurate model $\beta = 6$. Furthermore, it can be observed that none of the "*exp*" model performed as the most accurate individual for any of the examples at any reliability level.

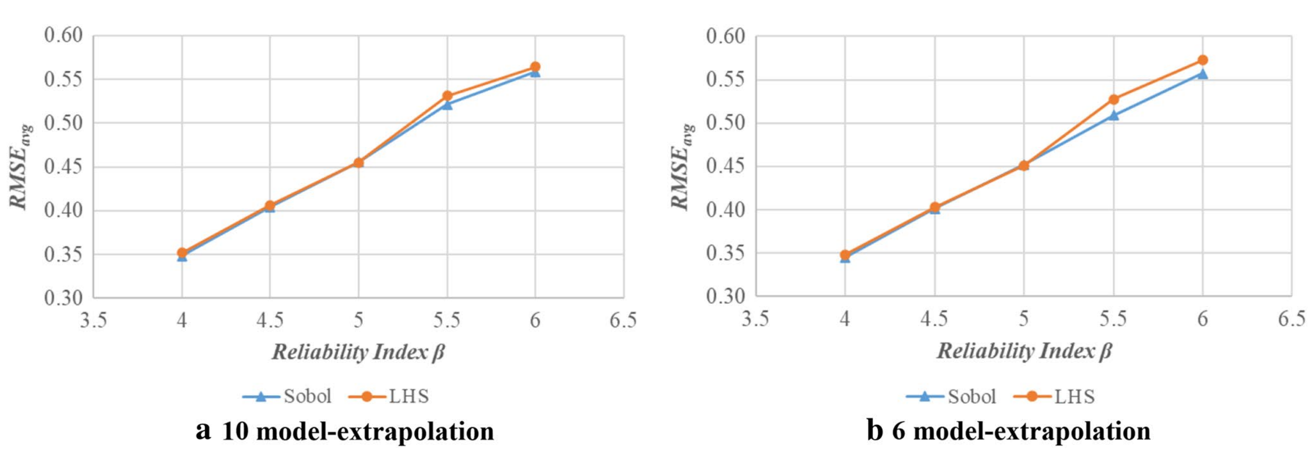

Fig. 9 The average RMSE values of LHS and Sobol sampling for different reliability levels

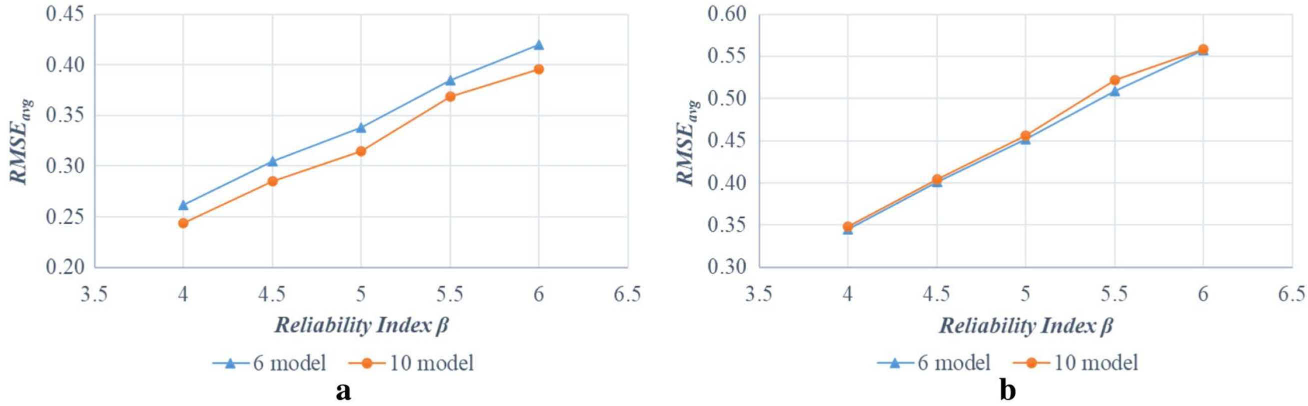

Fig. 10 RMSE values of **a** the best individual extrapolation models, and **b** the mean extrapolation models for the 6-model and 10-model extrapolation formulas for different reliability levels

Based on a comparison of the performances of the two mean extrapolation formulations, Fig. 10b shows that there is no strong reason to choose 10-model formulation over 6-model formulation. For the example problems investigated, it is observed that the 6-model mean extrapolation yields slightly more accurate results than the 10-model mean extrapolation (see Fig. 10b). It is discovered that adding more models to the mean extrapolation formulation does not necessarily improve the performance of the mean extrapolation prediction, because some individual models participating in the mean prediction deteriorates the performance of the extrapolation method. Because the 6-model mean extrapolation results in more accurate predictions than the 10-model mean extrapolation, we decide to carry out the subsequent studies using the 6-model mean extrapolation formulation.

Additionally, we compare the accuracies from using the mean and median values of multiple predictive models. Figure 11 shows that using the mean value of multiple predictive models yields lower $RMSE_{nor}$ values for both the 6-model and 10-model extrapolation formulations.

5.3 Effect of initial scale parameter

Another factor that substantially affects the performance of asymptotic sampling is the value of the initial scale parameter, f_0. The use of a smaller f_0 value causes an increase in the number of samples falling into the failure domain in the first steps of the asymptotic sampling process. Hence, the required number of samples in the failure domain is attained faster, and the *NFE* value used throughout the process decreases, which reduces the computational cost. For the cantilever beam problem at a reliability index value of 4.03, as a demonstration example, Fig. 12a shows that the number of limit state function evaluations, *NFE*, corresponding to $f_0 = 0.2$ is substantially smaller than the *NFE* corresponding to $f_0 = 1.0$. By contrast, the use of a smaller f_0 value results in a set of support points with smaller scaled reliability indices and leaves more room for extrapolation error. Figure 12a shows that the $RMSE_{nor}$ corresponding to $f_0 = 0.2$ is substantially larger than that corresponding to $f_0 = 1.0$. This behavior is similar for all example problems and reliability indices presented herein (see "Appendix B"). Therefore, the optimum f_0 value should be determined.

To determine the optimum value of f_0, we generate *NFE* versus $RMSE_{nor}$ plots for a wide range of f_0 values (we decrease f_0 from 1 to 0.2 at 0.1 intervals). We consider that $RMSE_{nor}$ values above 10% are not acceptable. We obtain the value of f_0 that gives acceptable $RMSE_{nor}$ values with the smallest *NFE*, and regard it as the optimum f_0. Figure 12b shows the *NFE* versus $RMSE_{nor}$ plots for $f_0 = 0.3$, $f_0 = 0.4$ and $f_0 = 0.5$. As shown, the initial scale parameter $f_0 = 0.4$ yields acceptable $RMSE_{nor}$ values with the smallest *NFE*; therefore, we regard it as the optimum value for this problem at this reliability level.

Table 9 lists the optimum f_0 values for all the example problems investigated. It is discovered that the optimum f_0 value ranges between 0.2 and 0.5. For reliability level $\beta = 4$, the average value of the optimum f_0 over all the problems considered is 0.4. Furthermore, it is also observed that as the reliability level increases, the optimum f_0 value decreases.

Next, we evaluate the effects of dimensionality and nonlinearity on the optimum initial scale parameter. For the reliability level of $\beta = 4$, Fig. 13 shows that the dimensionality does not have a linear relationship with the optimum value of f_0 (notice small R^2 values in the figures), whereas the nonlinearity has (notice significant R^2 values in the figures). As the nonlinearity of the limit state function increases, the optimum value of f_0 also increases. The effects of dimensionality and nonlinearity at the other reliability levels of (that is, $\beta = 4.5$, 5.0, 5.5) are explored in "Appendix C". For those reliability levels, the finding is the same that not have

Table 8 Optimum individual extrapolation models for different problems at different reliability levels

Rel. index	4		4.5		5		5.5		6	
Problem	6 model ext	10 model ext	6 model ext	10 model ext	6 model ext	10 model ext	6 model ext	10 model ext	6 model ext	10 model ext
CB	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_2$	$nor\ q_2$	$nor\ q_3$
CC	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$
CR	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$
FC	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$
IB	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$	$nor\ q_2$	$nor\ q_3$
RT	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_1$	$nor\ q_{0.5}$	$nor\ q_{0.5}$

a linear relationship with the optimum value of f_0, whereas the nonlinearity has, and that the optimum value of f_0 also increases as the nonlinearity of the limit state function increases.

5.4 Effect of number of support points

To improve the performance of asymptotic sampling, the appropriate number of support points, N_s, must be used. In this regard, Bucher (2009) suggested the use of five support points. However, in his later studies, he used different numbers of support points. In a follow-up study, Gasser and Bucher (2018) suggested using four or more support points to yield a more stable regression; however, this approach resulted in an increase in computational effort. They used different numbers of support points in different example problems without clarifying the process by which appropriate number is determined.

In this study, we obtain the optimum number of support points in a similar fashion as we performed in determining the optimum value of f_0. We generate *NFE* versus $RMSE_{nor}$ plots for N_s values ranging between 2 to 5, find the value of N_s that gives acceptable $RMSE_{nor}$ values with the smallest *NFE*, and regard it as the optimum N_s. Figure 14 shows that the optimum value of N_s is 4 for the cantilever beam problem at a reliability index of $\beta=4.03$.

Table 10 lists the optimum N_s values for all the examples presented herein. The optimum N_s value is 4 for most ıf the cases. Furthermore, as the reliability level increases, the N_s value should be increased to 5. In summary, for the moderate reliability index values (β=4, 4.5, and 5) N_s should be 4; however, at reliability levels with higher reliability index values (β=5.5, and 6) N_s should be increased to 5.

Next, we evaluate the effects of dimensionality and nonlinearity on the optimum number of support points. For reliability level $\beta=4$, Fig. 15a shows that neither the dimensionality nor the nonlinearity have an important effect on the optimum number of support points. On the other hand, for reliability level $\beta=6$, Fig. 15b shows that the optimum number of points increases as the nonlinearity increases, even though the dimensionality is still not effective. The effects of dimensionality and nonlinearity at the other reliability levels (i.e., $\beta=4.5$, 5.0, and 5.5) are explored in "Appendix C". The observations at $\beta=4.5$ and $\beta=5.0$ are the same as those of $\beta=4.0$, and the observations at $\beta=5.5$ are the same as those of $\beta=6.0$.

5.5 Application to a complex problem

The limit state functions for the example problems presented thus far involve functional expression between the input and output parameters. However, the asymptotic sampling method does not impose such a restriction. Next, we present

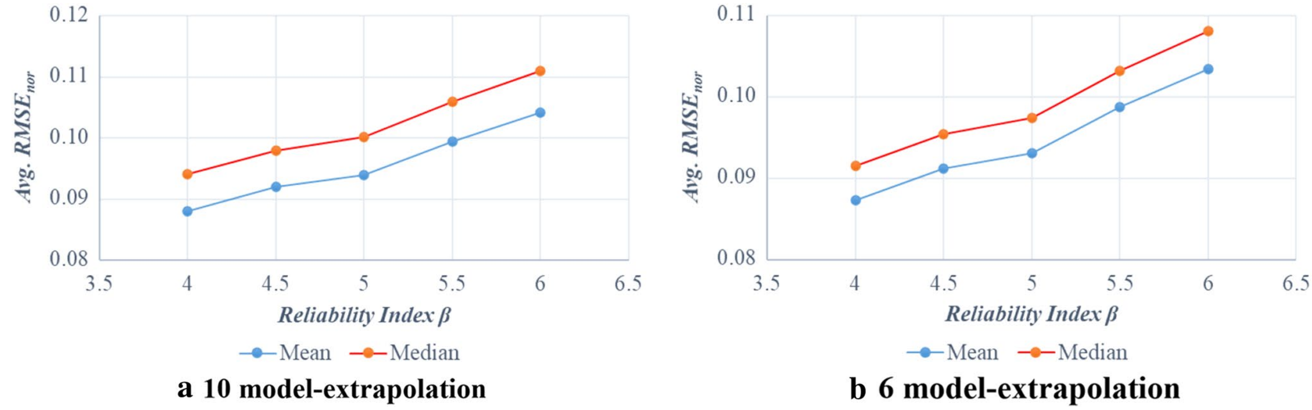

Fig. 11 The average $RMSE_{\text{nor}}$ values of mean and median of the multiple predictive models

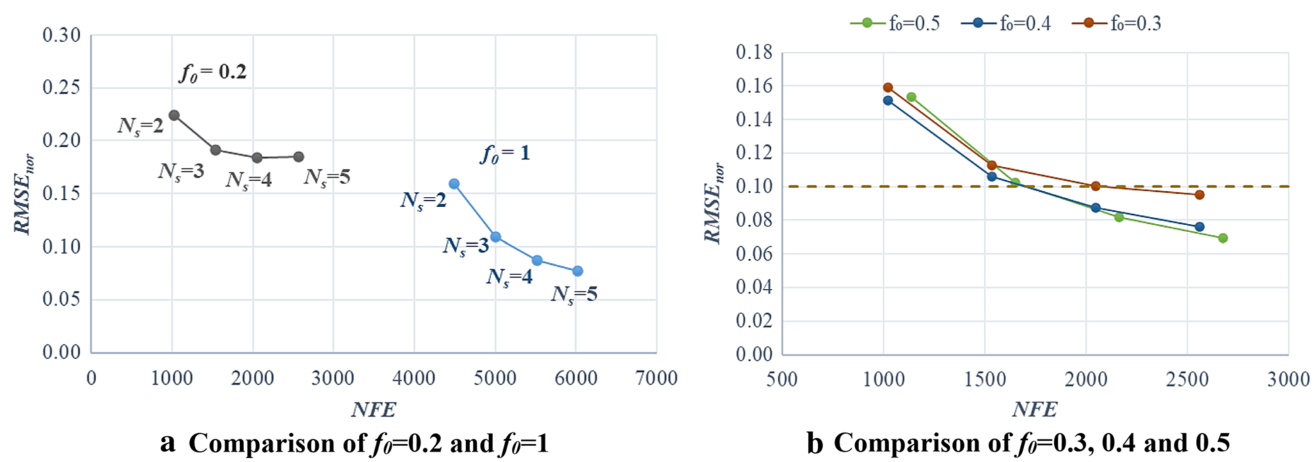

Fig. 12 Comparison of performances of the asymptotic sampling method for different values of f_0 for cantilever beam problem for reliability index of $\beta = 4.03$

Table 9 Optimum f_0 values for all example problems at various reliability levels

Problem	Rel. index				
	4	4.5	5	5.5	6
CB	0.4	0.4	0.4	0.4	0.3
CC	0.4	0.3	0.3	0.3	0.2
CR	0.3	0.3	0.3	0.3	0.2
FC	0.5	0.4	0.4	0.4	0.3
IB	0.4	0.4	0.3	0.3	0.3
RT	0.4	0.4	0.4	0.4	0.4
Average	0.4	0.37	0.35	0.32	0.28

an example to demonstrate the application of the asymptotic sampling method to a complex problem without a functional expression between the input and output parameters.

The design of an automobile torque arm is presented in this section. This problem was originally presented by Bennett and Botkin (1986) and has been investigated by several researchers including Picheny et al. (2008), Rahman and Wei (201), and Acar (2011). Briefly, the torque arm is subject to a horizontal load $F_x = -2789$ N, and a vertical load $F_y = 5066$ N (see Fig. 16). The loads are transmitted from a shaft at the right hole, and the left hole is fixed. The torque arm material has Young's modulus of $E = 206.8$ GPa, and Poisson's ratio of $\nu = 0.29$. Seven design variables (d_1 through d_7) alter the shape of the torque arm as shown in Fig. 17.

The limit state function for the torque arm problem is formulated as

$$Y = \sigma_f - \sigma_{\max} \tag{20}$$

where σ_f is the failure stress of the torque arm material and $\sigma_{\max}$ is the maximum von Mises stress developed at the torque arm. The stresses at the control arm are computed through finite element analysis using a MATLAB finite

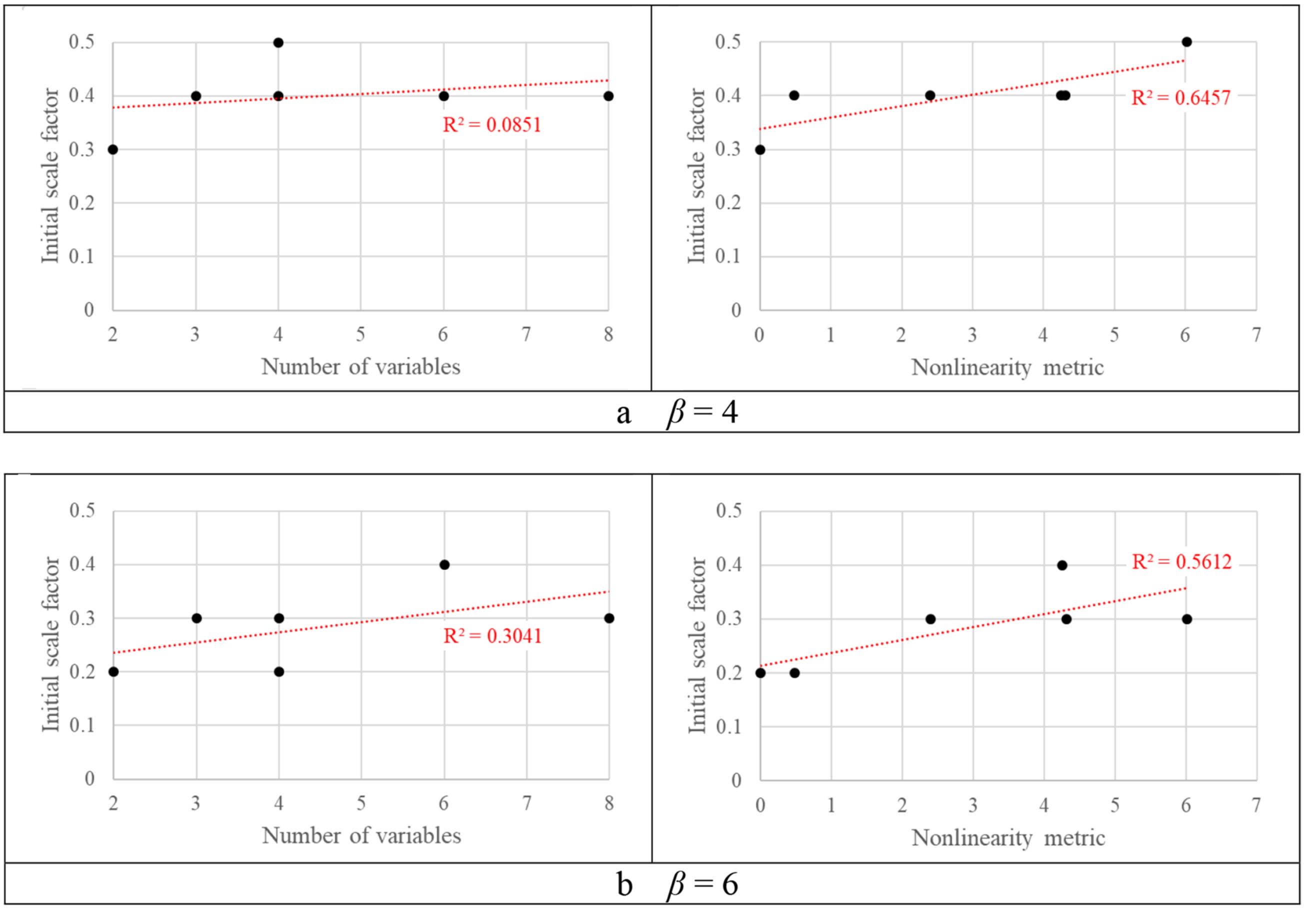

Fig. 13 The effects of dimensionality and nonlinearity on the optimum value of f_0

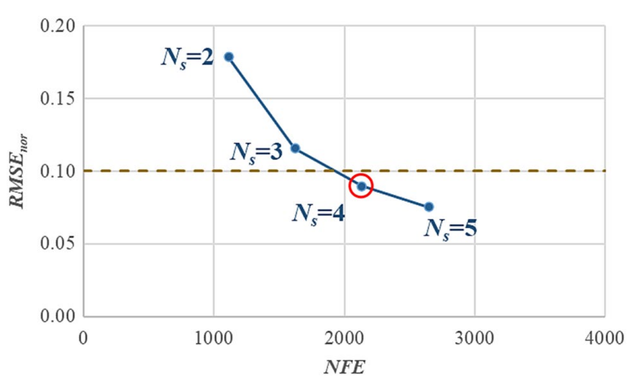

Fig. 14 Comparison of performances of the asymptotic sampling method for different N_s values for CB problem for reliability index $\beta = 4.03$, and $f_0 = 0.4$

Table 10 Optimum N_s values for all example problems

Problem	Rel. index				
	4	4.5	5	5.5	6
CB	4	4	4	5	5
CC	4	4	4	4	4
CR	4	4	4	4	4
FC	4	4	4	5	5
IB	4	4	4	4	4
RT	4	4	4	4	5

element toolbox developed by Maute (2009) and CALFEM (1999). Figure 18 depicts the von Mises stress distribution on the torque arm when the design variables and the applied loads are assigned to their mean values. All the seven design variables (d_1 through d_7), applied loads (F_x and F_y), and failure stress are regarded as random variables. The statistical properties of the random variables are presented in Table 11. For the loads, a load safety factor of $n_L = 3$ is used to maintain a sufficient reliability level.

The reliability estimation of the torque arm is performed using asymptotic sampling, where the Sobol sequence is used in sampling, 6-model mean extrapolation formulation is adopted, the initial scale parameter is chosen as 0.4, and four support points are used. For this problem, the asymptotic sampling process is repeated 100 times (as opposed to 1000 times for the earlier examples). The reliability index of the torque arm is also estimated using MCS to have a basis

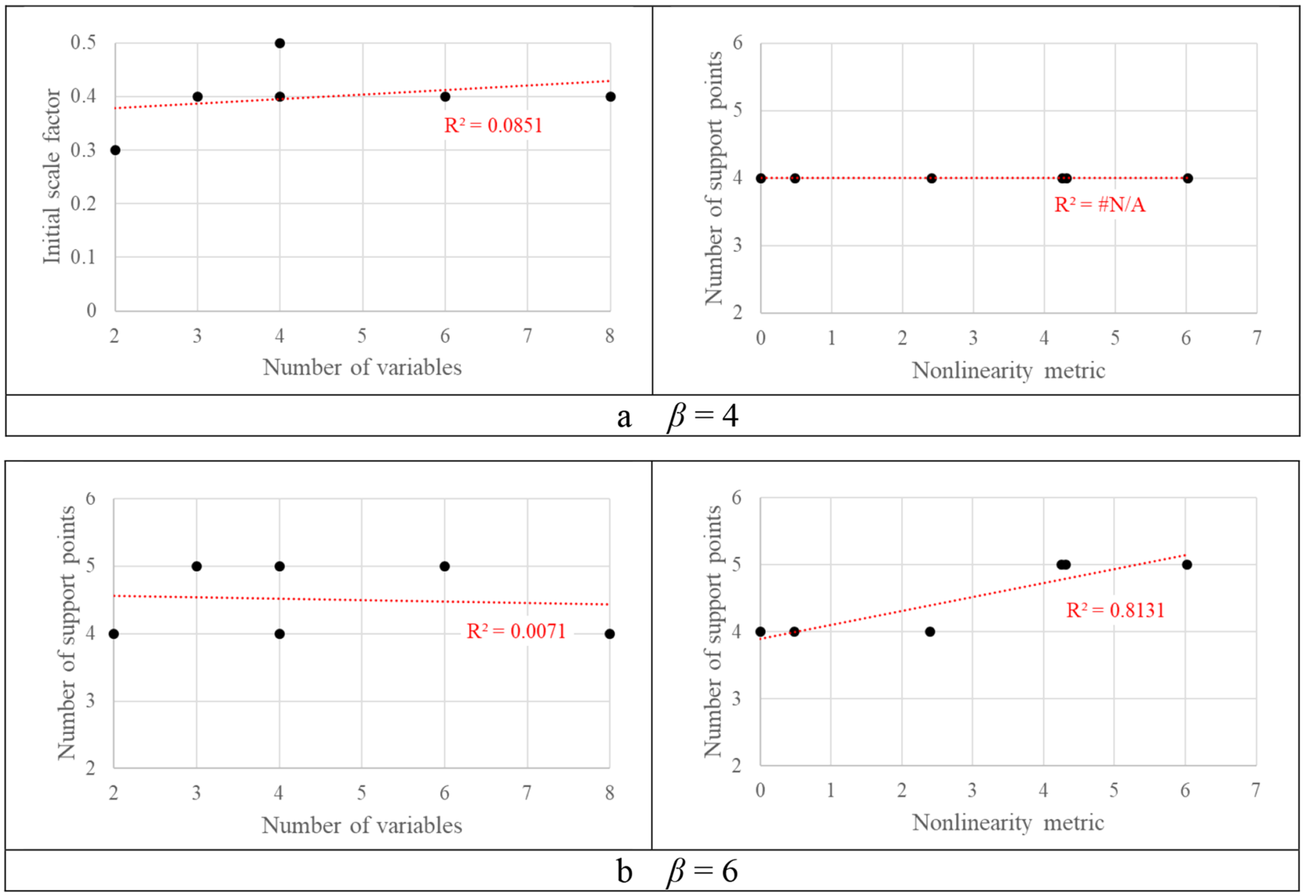

Fig. 15 The effects of dimensionality and nonlinearity on the optimum number of support points N_s

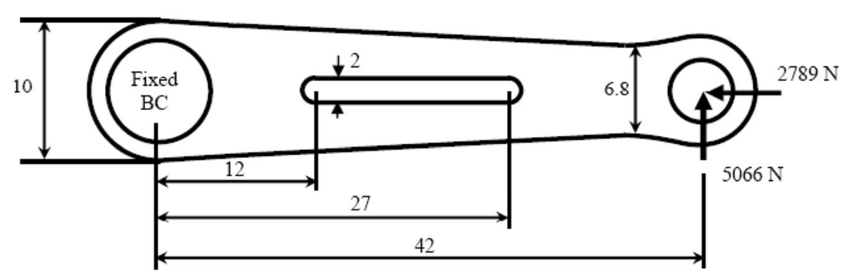

Fig. 16 The loading and boundary conditions for the torque arm. Dimensions are in cm. [Courtesy of Picheny et al. 2008]

for comparison. A sample size of 10^6 is used, and the reliability index is computed as 4.17. Table 12 shows that the reliability index is estimated to be 4.21 on average, which is similar to that estimated through MCS (with approximately 1% error), and the $RMSE_{\mathrm{nor}}$ value is estimated to be 9.2%. Furthermore, Table 12 shows that if the number of support points is increased to 5, then the NFE will increase from 2048 to 2560, without a significant accuracy gain.

6 Concluding remarks

The performance of the asymptotic sampling method depends on various factors including the sampling method, formulation of extrapolation models, initial scale parameter, and number of support points. To analyze the effects of these factors, the asymptotic sampling method with different settings of these factors was applied to various

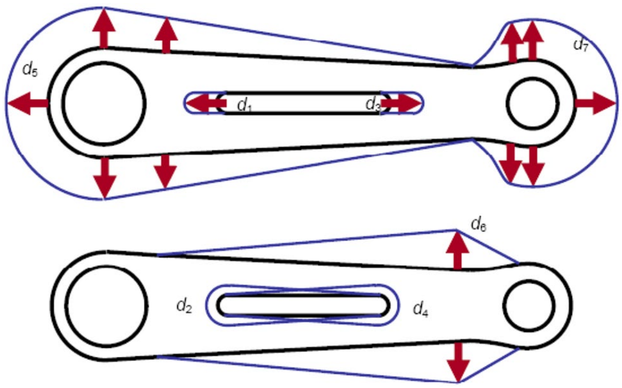

Fig. 17 Design variables used to alter the shape of the torque arm. [Courtesy of Picheny et al. 2008]

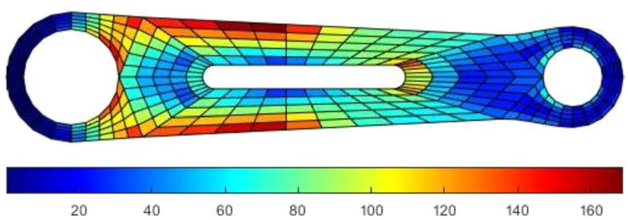

Fig. 18 Von Mises stress distribution on the torque arm when the design variables and the applied loads are assigned to their mean values. Stresses are in MPa

Table 11 The statistical properties of the random variables for the torque arm problem. A load safety factor of $n_L = 3$ is used

Random variable	Distribution type	Mean; standard deviation
d_1 through d_7	Normal	0; 0.1
F_x	Normal	$-2789/n_L$; $278.9/n_L$
F_y	Normal	$5066/n_L$; $506.6/n_L$
S	Lognormal	300; 30

Table 12 The performance of the asymptotic sampling for the torque arm problem

Number of support points	4	5
Number of function evaluations	2048	2560
Rel. index prediction (average over 100 runs)	4.21	4.23
$RMSE_{nor}$	0.092	0.084

structural mechanics example problems with varying numbers of random variables and probability distribution types, and the following conclusions were drawn:

- First, the effect of the sampling method on the asymptotic sampling performance was analyzed. The use of LHS and Sobol sequence was compared, and it was found that there was no remarkable difference between the sampling methods. However, the use of Sobol sequences yielded slightly smaller prediction errors at higher reliability levels.
- Next, we compared the 6-model and 10-model mean extrapolation formulations. We discovered that the 6-model mean extrapolation formulation was more accurate than the 10-model mean extrapolation formulation. This finding implied that adding more models to the mean extrapolation formulation did not necessarily improve the performance of the mean extrapolation prediction.
- Subsequently, the effect of the initial scale factor was explored. It was find that the initial scale parameter could be set between 0.3 and 0.4 for a reliability index range of 4–6. As the reliability level increased, the initial scale parameter should be decreased. For all reliability levels, it was found that the dimensionality did not have a linear relationship with the optimum f_0 value, whereas the nonlinearity had. It was also observed that the optimum f_0 value also increased as the nonlinearity of the limit state function increased.
- Next, the effect of the number of support points was explored. It was discovered that the use of 4 support points provided the best compromise between accuracy and efficiency. Meanwhile, if the reliability index is extremely high, then the use of 5 support points is recommended to achieve an acceptable level of accuracy. For reliability indices 4, 4.5 and 5.0, it was observed that neither the dimensionality nor the nonlinearity had an important effect on the optimum number of support points. On the other hand, for larger reliability indices, the optimum number of support points increased as the nonlinearity increased, even though the dimensionality was still not effective.
- Finally, the performance of the asymptotic sampling was tested on a complex problem that was similar to real-life problems. The Sobol sequence was used for sampling, 6-model mean extrapolation formulation was adopted,

the initial scale parameter was set to 0.4, and four support points were used. It was found that these settings provided a good compromise between accuracy and efficiency.

Some limitations of the current study are as follows:

- The asymptotic sampling method is based on the assumption that the reliability index increases monotonically as the standard deviation scale factor increases. For problems with multiple failure modes and multiple MPPs, this assumption may not be applicable. Hence, this method is not applicable to these abovementioned problems.
- The asymptotic sampling method works for non-normal distribution types provided that the distributions of interest are not described by higher moments.
- Even though the suggested settings yielded good results for the complex application problem (i.e., torque arm problem), the number of example problems should be increased to further validate these settings.

Future research could focus on the following subjects:

- In the mean extrapolation formulations, a simple averaging is used. A better strategy is to use a weighted average formulation rather than simple averaging. The accuracy of each individual extrapolation model can be estimated, and the weight factors in the weighted average model could be selected such that the accurate models have larger weight factors.
- Asymptotic sampling can be combined with importance sampling to further increase its efficiency.
- In the numerical examples tested, the reliability levels are changed by scaling the constraint bounds. The effect of the standard deviations on the results is worth investigating in the future.

Appendix A: Reliability levels of the numerical example problems

For all numerical example problems, five different reliability levels are considered by changing a proper term in the LSF (see Table 13). The reliability index values reported in Table 13 are predicted using crude Monte Carlo simulations with a sample size of 10^7, 10^8, 10^9, 10^{10}, and 10^{11} for reliability indices of 4, 4.5, 5, 5.5, and 6. Note that the reliability indices of 4, 4.5, 5, 5.5, and 6 correspond to the failure probabilities of 3.17×10^{-5}, 3.40×10^{-6}, 2.87×10^{-7}, 1.90×10^{-8}, and 9.87×10^{-10}, respectively.

Appendix B: NFE versus RMSE plots for all example problems and all reliability levels

The *NFE* values corresponding to different values of f_0 for all reliability levels of the example problems are provided in Figs. 19, 20, 21, 22, 23 and 24. It can be realized that the $RMSE_{\text{nor}}$ values are greatly increased when $f_0=0.2$ at all reliability levels for all example problems. For this reason, we did not investigate the values of f_0 below 0.2.

Appendix C: The effects of dimensionality and nonlinearity on the optimum initial scale parameter and the number of support points

In this appendix, the effects of dimensionality and nonlinearity on the optimum value of the initial scale parameter f_0 and the optimum number of support points N_s for the reliability levels $\beta=4.5$, 5.0, and 5.5 are explored. Note that the effects in the reliability levels $\beta=4.0$ and 6.0 are explored in the main text, in Sects. 5.3 and 5.4.

Figure 25 shows for all reliability levels that the dimensionality does not have an important effect on the optimum

Table 13 The reliability levels considered for the example problems

ID	Problem	Term[a]	Value[b]	β[c]	Value[b]	β[c]	Value[b]	β[c]	Value[b]	β[c]	Value[b]	β[c]
1	Connection rod	μ_R	60	4.00	55	4.50	50	5.00	45	5.50	40	6.00
2	Cantilever beam	D_0	2.50	4.03	2.62	4.51	2.75	5.00	2.89	5.54	3.04	6.05
3	Central crack	$\overline{K}_{IC}$	52	4.01	57	4.52	63	5.01	70	5.52	79	6.04
4	Fortini's clutch	y_{crit}	4.05	4.02	3.55	4.53	3.02	5.01	2.31	5.50	1.20	6.04
5	Roof truss	c	0.0360	4.07	0.0378	4.53	0.0400	5.01	0.0425	5.50	0.0466	6.07
6	I beam	$\overline{S}$	410×10^3	4.07	490×10^3	4.50	630×10^3	5.01	880×10^3	5.49	1700×10^3	6.06

[a]The term in the limit state function that is varied to change the reliability level

[b]The value of the term

[c]Corresponding reliability index

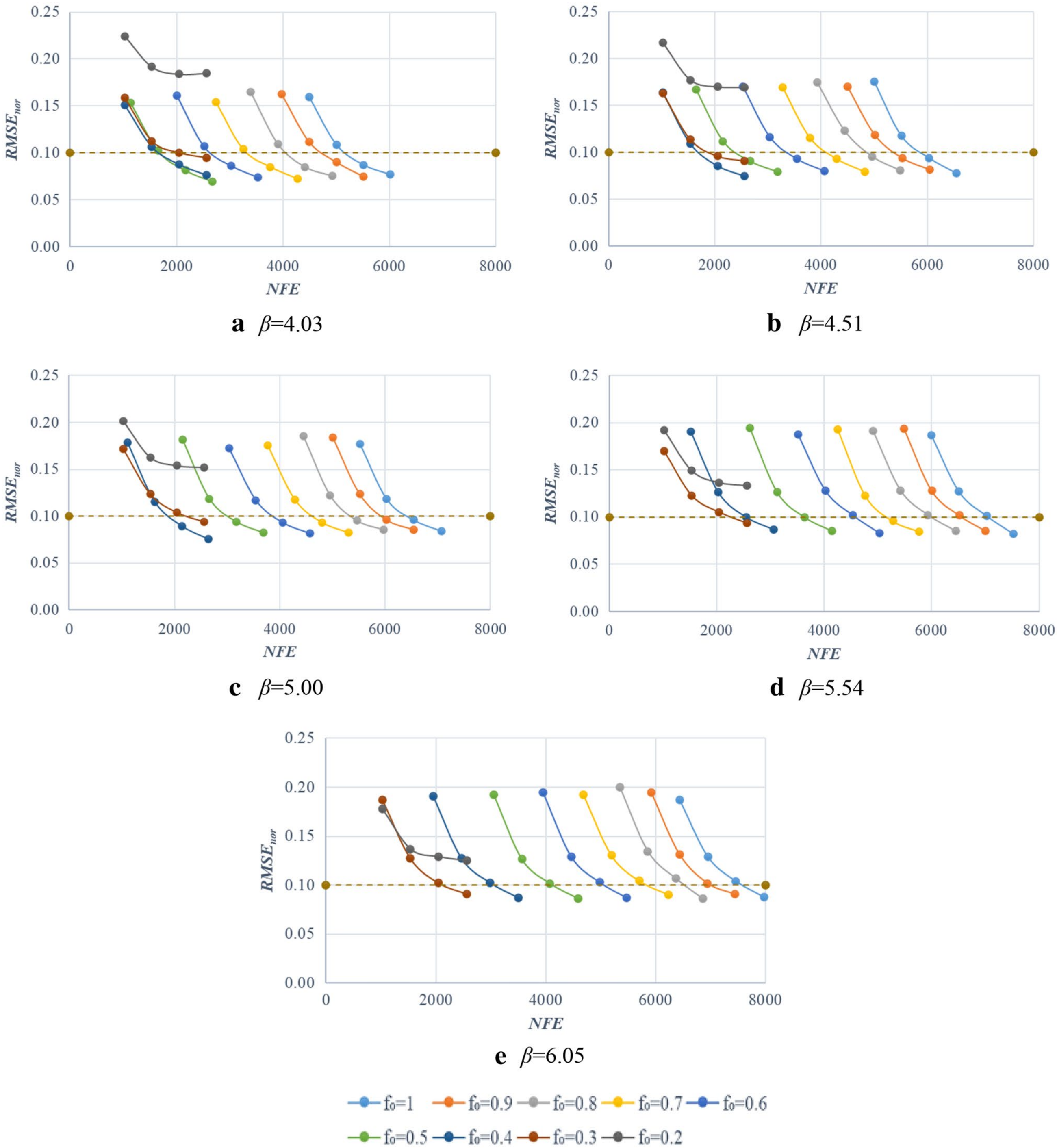

Fig. 19 *NFE* values corresponding to f_0 values for cantilever beam problem

value of f_0, whereas the nonlinearity has a substantial effect. As the nonlinearity of the limit state function increases, the optimum value of f_0 also increases.

Figure 26 shows for the reliability levels $\beta = 4.5$ and $\beta = 5.0$ that neither dimensionality nor nonlinearity have an important effect on the optimum number of support points N_s. However, for the reliability level $\beta = 5.5$, it is seen that

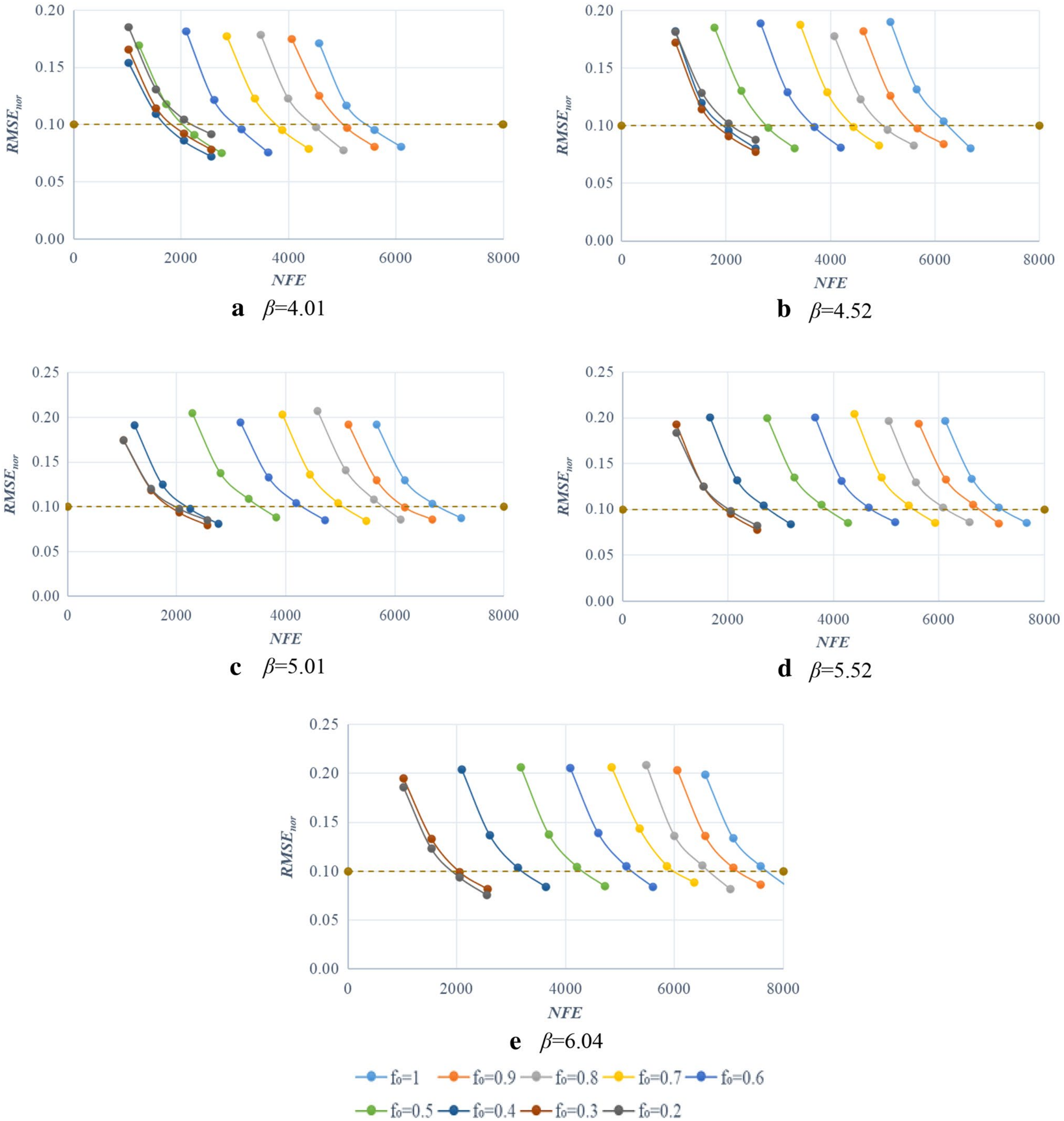

Fig. 20 *NFE* values corresponding to f_0 values for central crack problem

the optimum number of support points increases as the nonlinearity of the LSF increases, even though the dimensionality is still insignificant.

Acknowledgements This paper is written to acknowledge Prof. Raphael (Rafi) T. Haftka's novel contributions in the field of structural and multidisciplinary design optimization. Rafi was a pioneer in our optimization community and had contributed significantly to the fields of reliability-based design optimization, surrogate-based optimization, structural and multidisciplinary optimization, sensitivity analysis,

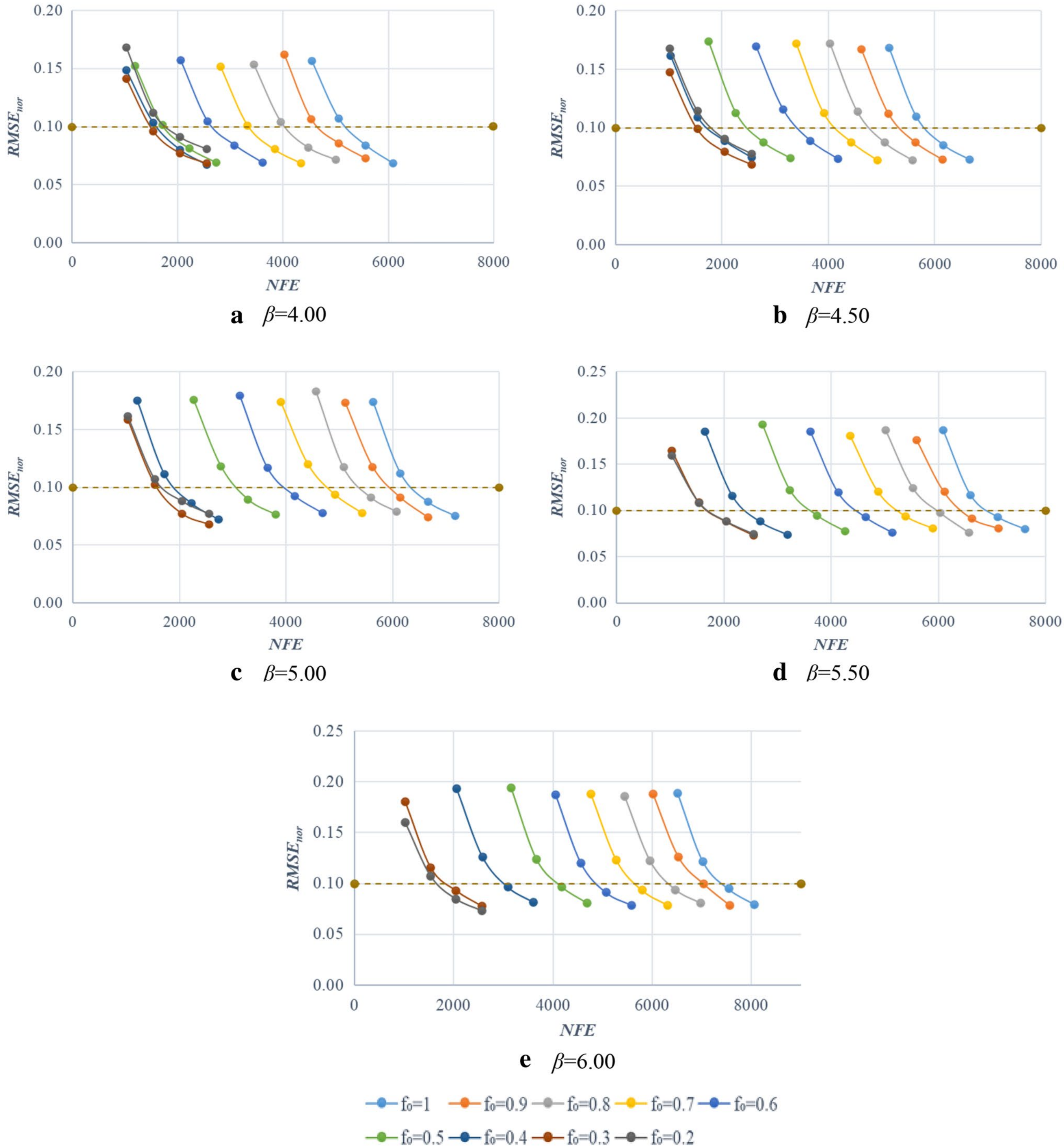

Fig. 21 *NFE* values corresponding to f_0 values for connection rod problem

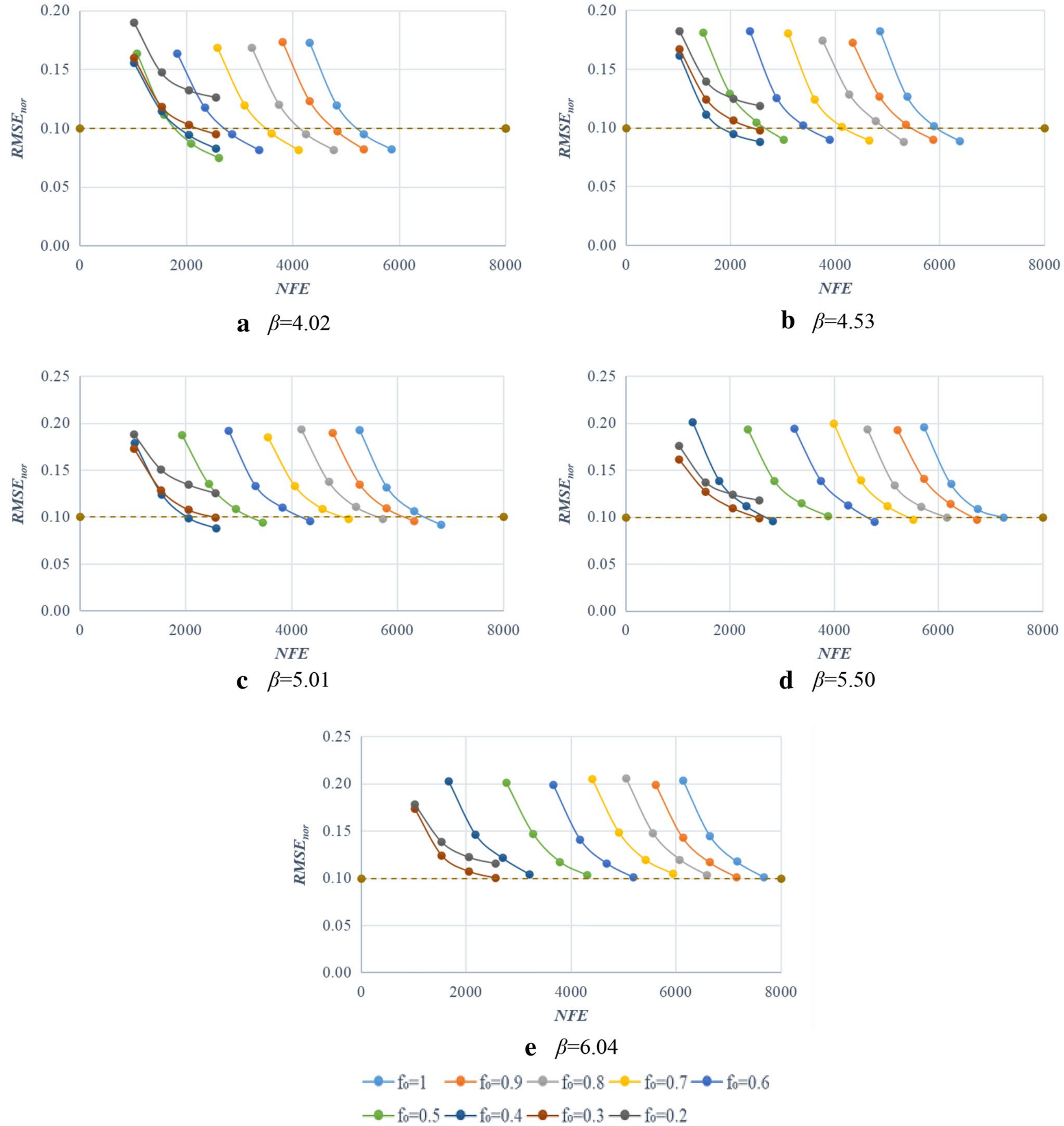

Fig. 22 *NFE* values corresponding to f_0 values for Fortini's clutch problem

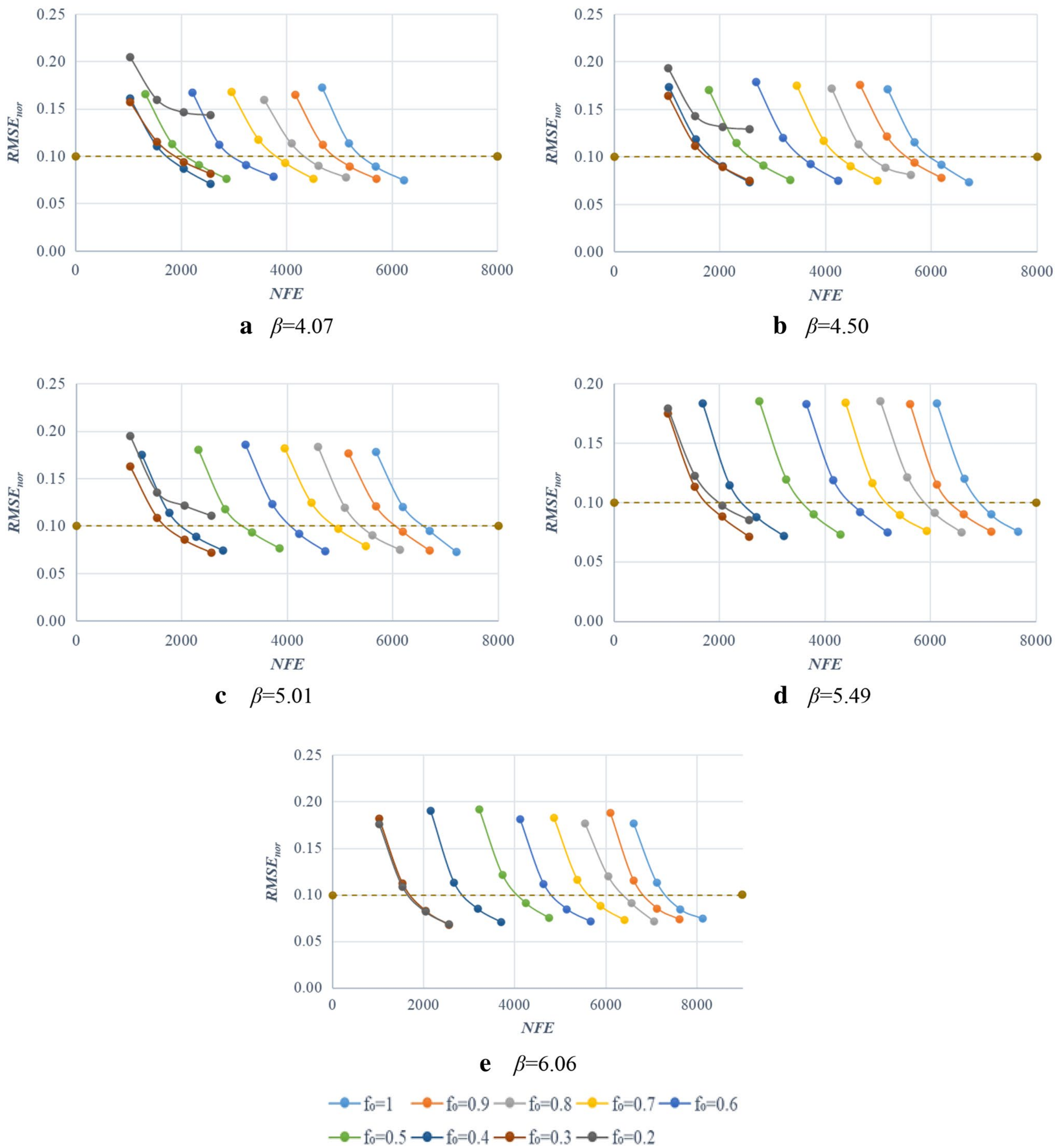

Fig. 23 *NFE* values corresponding to f_0 values for I beam problem

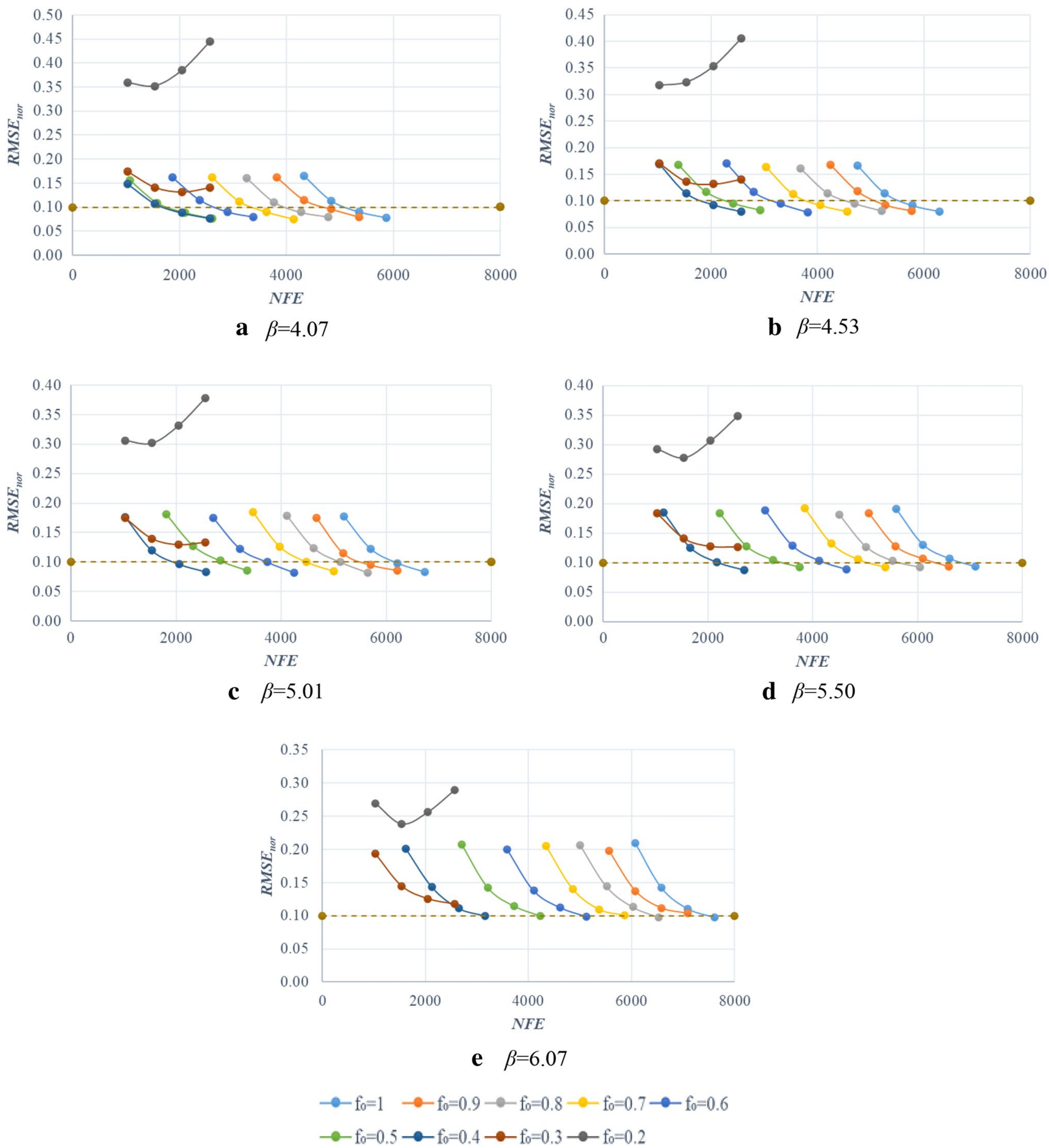

Fig. 24 *NFE* values corresponding to f_0 values for roof truss problem

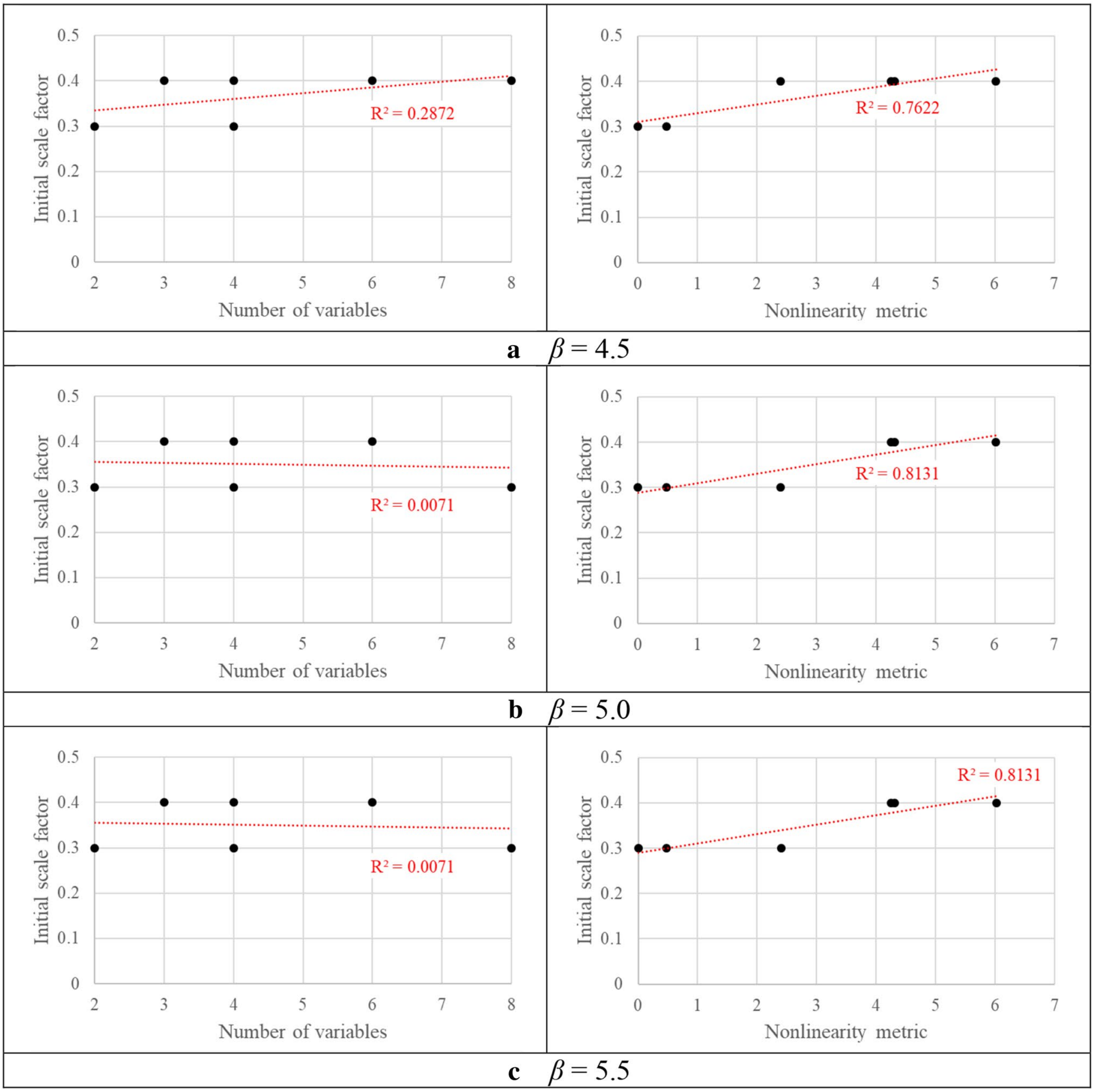

Fig. 25 The effects of dimensionality and nonlinearity on the optimum f_0 value for reliability levels $\beta = 4.5$, 5.0, and 5.5

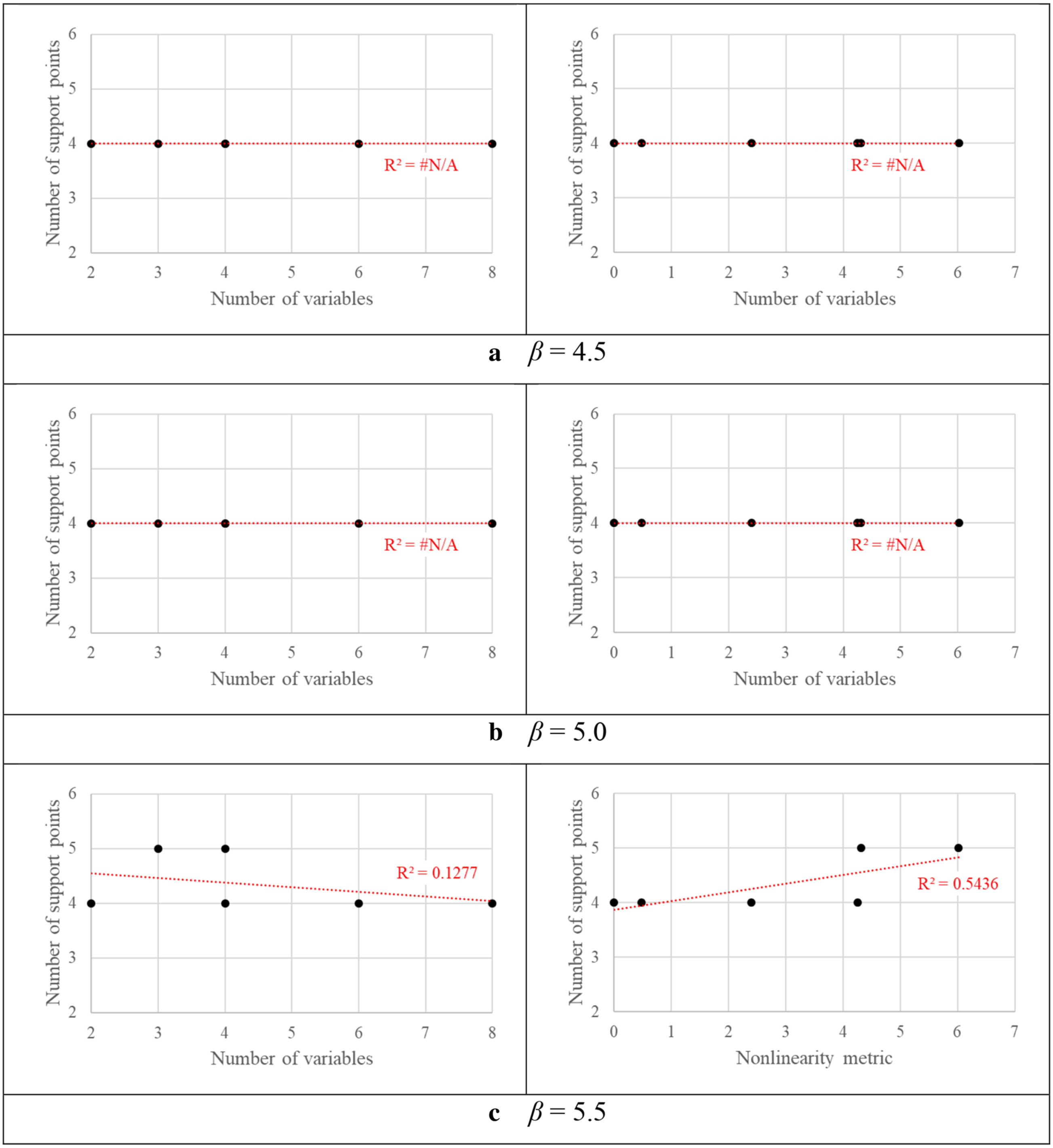

Fig. 26 The effects of dimensionality and nonlinearity on the optimum N_s value for reliability levels $\beta=4.5$, 5.0, and 5.5

optimization of laminated composite materials among others. The corresponding author is a PhD student of Rafi, and he is grateful for the guidance and camaraderie of him.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Replication of results The results provided herein are replicable. Interested readers can contact the corresponding author to obtain the MATLAB codes used to generate the results.

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