JOURNAL OF AIRCRAFT
Vol. 44, No. 3, May–June 2007

Reliability-Based Aircraft Structural Design Pays, Even with Limited Statistical Data

Erdem Acar* and Raphael T. Haftka†
University of Florida, Gainesville, Florida 32611-6250

DOI: 10.2514/1.25335

Probabilistic structural design tends to apply higher safety factors to inexpensive or lightweight components, because it is a more efficient way to achieve a desired level of safety. We show that even with limited knowledge about stress probability distributions, we can increase the safety of an airplane by following this paradigm. We find that a small perturbation of the deterministic design is sufficient to maximize safety for the same weight. The structural optimization for safety of a representative system composed of a wing, a horizontal tail, and a vertical tail is used to demonstrate the paradigm. We find that moving a small amount of material from the wing to the tails leads to substantially increased structural safety. Because aircraft companies often apply additional safety factors beyond those mandated by the Federal Aviation Administration, this opens the door to obtaining probabilistic designs that also satisfy the Federal Aviation Administration code-based rules for deterministic design. We also find that the ratio of probabilities of failure of the probabilistic design and the deterministic design is insensitive to even very large errors in the stress probability distribution or probability-of-failure estimate of the deterministic design. Finally, we find that for independent components subject to the same failure mode, the probabilities of failure at the probabilistic optimum are approximately proportional to the weight. So a component that is 10 times lighter than another should be designed to be about 10 times safer. This phenomenon is proved for normal distributions of stress and failure stress, but was found in an example to approximately hold also for the lognormally distributed stress.

Nomenclature

b	=	reliability index
$F()$	=	cumulative distribution function of the failure stress
$f()$	=	probability density function of the failure stress
k	=	proportionality constant for the relative changes in stress and relative change in characteristic stress
P_f	=	actual probability-of-failure probabilistic design
P_f^*	=	approximate probability-of-failure of probabilistic design
P_{fd}	=	probability-of-failure of deterministic design
$s()$	=	probability density function of the stress
W_d	=	weight of deterministic design
W	=	weight of probabilistic design
Δ	=	relative change in stress
Δ^*	=	relative change in characteristic-stress σ^* corresponding to a relative change of Δ in stress σ
μ_f	=	mean value of the failure stress
μ_σ	=	mean value of the stress
σ	=	stress
σ_f	=	failure stress
σ^*	=	characteristic stress

Subscripts

d	=	deterministic
W and T	=	wing and tail, respectively

Presented as Paper 2059 at the 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Newport, RI; received 20 May 2006; revision received 16 January 2007; accepted for publication 19 January 2007.

*Ph.D., Mechanical and Aerospace Engineering Department, currently Postdoctoral Associate at the Center for Advanced Vehicular Systems, Mississippi State University, Starkville, Mississippi 39762; erdem@cavs.msstate.edu. Student Member AIAA.

†Distinguished Professor, Mechanical and Aerospace Engineering Department; haftka@ufl.edu. Fellow AIAA.

I. Introduction

AEROSPACE structures have traditionally been designed using a deterministic (or code-based) approach based on the Federal Aviation Administration (FAA) regulations. Structural safety has been achieved by combining safety factors with tests of material and structural components. In design of transport aircraft, the FAA requires the use of a safety factor of 1.5 for loads and conservative material properties (A-basis value or B-basis value, depending on the failure path) to maintain a high level of safety for the aircraft. In addition, companies add conservative design measures to insure that they do not fail certification. Acar et al. [1,2] analyzed the safety measures and found that the load safety factor of 1.5 complemented with conservative material properties, redundancy, and certification testing raises the actual safety factor to about 2.

The FAA design code is based on uniform safety factors (that is, the same safety factor is used for all components). Probabilistic design derives an important part of its advantage over deterministic design by allowing the use of substantially nonuniform safety factors; hence, there is growing interest in replacing safety factors by probabilistic design (e.g., Lincoln [3], Wirsching [4], a Society of Automotive Engineers report [5], and Long and Narciso [6]). However, with only partial information on statistical distributions of variabilities, and guesswork on reasonable distributions for errors, engineers are reluctant to pursue probabilistic design. It has also been shown that insufficient information may lead to large errors in probability calculations (e.g., Ben-Haim and Elishakoff [7] and Neal, et al. [8]). The main objective of this paper is to show that we can increase the safety of an airplane without increasing its weight even with the limited data available today following the design paradigm mentioned (higher safety factors for lightweight components). Our approach uses two statistical data that are well understood. The first is the statistical distribution of failure stress, which is required by the FAA for choosing A-basis or B-basis allowables. The second is a special property of the normal distribution: when a large number of uncertainties contributes to a distribution, it tends to become similar to a normal distribution. This applies to the stress estimation because it is influenced by a large number of error and variability sources. Finally, we show that even though the limited statistical data may substantially affect the probabilities of failure of both the probabilistic design and the code-based deterministic design, the ratio of probabilities of failure of the probabilistic design and the deterministic design is insensitive to even large errors, due to limited data.

The paper is structured as follows. Reliability-based design optimization (RBDO) of a representative wing and tail system with perfect and limited statistical data is given in Sec. II. Section III discusses the effects of errors in statistical information for the deterministic design. Section IV proposes an approximate method that allows probabilistic design based only on probability distribution of failure stresses. The application of the method to the wing and tail system, discussed earlier in Sec. II, is presented in Sec. V. Finally, the paper culminates with Sec. VI, in which the concluding remarks are listed.

II. Demonstration of Gains from RBDO of a Representative Wing and Tail System

A. Problem Formulation and Simplifying Assumptions

Calculating the probability of stress failure in a structure can be done by generating the probability distribution functions (PDF) $s(\sigma)$ of the stress and the PDF $f(\sigma_f)$ of the failure stress (see the solid lines in Fig. 1). Once these distributions are available, calculating the probability of failure can be accomplished by simple integration, as discussed later in the paper. The distribution of failure stress is typically available from experiments and so it does not require much computation. For materials used in aircraft design, the FAA regulations mean that statistical information on failure stresses is often available quite accurately. On the other hand, the PDF for the stress requires data such as analysis error distributions that are difficult to estimate, and it also requires expensive finite element computations. Fortunately, though, the PDF of the stress contains contributions from large number of parameters such as variabilities and errors in material properties and geometry and loading, and so it is likely to be well-represented by a normal distribution. However, estimating well the mean and standard deviation of that normal distribution is difficult due to limited data. In Sec. II.B, we momentarily disregard this difficulty to demonstrate the advantage of probabilistic design deriving from the use of higher safety factors for lighter structural components. In Sec. III, the effects of limited statistical data will be addressed.

We consider a representative wing and tail system. In general, to perform reliability-based design, we need to recalculate the stress PDF as we change the design. For the sake of simplicity, we assume that structural redesign changes the entire stress distribution, as shown in Fig. 1, by a simple scaling of σ to $\sigma(1+\Delta)$. This assumption will be accurate when the uncertainties are in the loading and the relative errors in stress calculation are not sensitive to the redesign.

We denote the failure probabilities of wing and tail obtained from deterministic design by $(P_{fW})_d$ and $(P_{fT})_d$, respectively. If failure of the two components is uncorrelated, the probability that at least one of them will fail is

$$P_{f0} = 1 - [1 - (P_{fW})_d][1 - (P_{fT})_d] \tag{1}$$

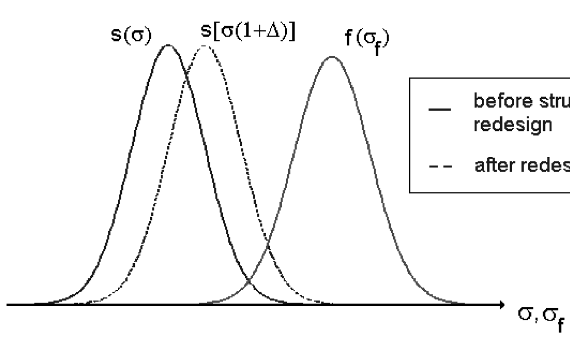

Fig. 1 Stress distribution $s(\sigma)$ before and after redesign in relation to failure-stress distribution $f(\sigma_f)$. We assume that redesign scales the entire stress distribution.

If the two failure probabilities are correlated, the calculation is still simple for a given correlation coefficient. For the purpose of demonstration, we make a simplifying assumption that the critical stress in each component is inversely proportional to weight. The term critical stress denotes the stress measure in the component that is associated with failure when it exceeds the failure limit. To simplify terminology, we omit the "critical" in the rest of the paper, so that we discuss stress versus failure stress.

The preceding assumption allows us to perform the demonstration without resorting to finite element modeling and analysis of these two components. That is, denoting the stresses in the wing and the tail by σ_W and σ_T, respectively, we use

$$\sigma_W = \frac{W_{dW}}{W_W}\sigma_{dW}, \qquad \sigma_T = \frac{W_{dT}}{W_T}\sigma_{dT} \tag{2}$$

where W_W and W_T are the wing and the tail structural weights, respectively, and the subscript d denotes the values of stresses and structural weights for the deterministic design. Given the distributions of σ_{dW} and σ_{dT} and the distribution of failure stresses Eq. (2) allows us to calculate the probability of failure of the wing for a given W_W and the probability of the failure of the tail for a given W_T. We can now formulate the following probabilistic design problem to minimize the probability of failure for constant weight

$$\min_{W_W, W_T} P_f(W_W, W_T) = 1 - [1 - P_{fW}(W_W)][1 - P_{fT}(W_T)]$$
$$\text{such that } W_W + W_T = W_{dW} + W_{dT} \tag{3}$$

The optimization problem stated in Eq. (3) is solved using the `fmincon` function of MATLAB that uses sequential quadratic programming. In the following subsection, we first perform probabilistic optimization for safety of the wing and tail system. Next, the effect of adding a vertical tail to the wing and horizontal tail system on the overall safety enhancement will be explored.

B. Probabilistic Optimization with Correct Statistical Data

For a typical transport aircraft, the structural weight of the horizontal tail is about 20% of that of the wing. So the weights of the wing and the tail before probabilistic optimization are taken as 100 and 20 units, respectively. We assume that the wing and tail are built from the same material and that the failure stress of the material follows lognormal distribution with a mean value μ_f of 100 and a coefficient of variation (COV) c_f of 10%. The COV of the stresses in the wing and the tail, c_σ, is assumed to be 20%. This may appear large in that stress calculation is quite accurate. However, there is substantial uncertainty in loading and geometry changes due to damage. For illustrative purpose, we assume that the historical record showed that the wing had a lifetime probability of failure of 1×10^{-7}. Because the deterministic design uses uniform safety factors (that is, the same safety factor is used for all components), and we assume that the wing and tail are made from the same material and have the same failure mode (point stress failure here), it is reasonable to assume that the probability of failure for the deterministic design of the tail is also 1×10^{-7}. As indicated earlier, we assume that the stresses follow normal distribution, which is characterized by only two parameters, mean and standard deviation. Therefore, given full information on the failure-stress distribution (lognormal with $\mu_f = 100$ and $c_f = 10\%$), the probability of failure ($P_f = 10^{-7}$), and the COV of the stress (c_σ=20%), the mean stresses in the wing and the tail are calculated as 39.77.[‡] The reader is referred to Appendix A for details of calculation of the unknown mean stresses in the wing and the tail for the given probability of failure and COV of the stresses.

[‡]For verification, we calculated the failure probability using conditional cumulative distribution function (CDF) Monte Carlo simulations of 10 million sample size for ten different times. The mean and standard deviation of these ten P_f calculations were 1.002×10^{-7} and 0.043×10^{-7}, respectively, showing that the probability-of-failure calculation explained in Appendix A is accurate.

Table 1 Probabilistic structural design optimization for safety of a representative wing and tail system. In the optimization, only the mean stresses are changed; the COV of the stresses are fixed at a COV of 0.20. Probability of failure of the wing and the tail for deterministic design are both 1×10^{-7}.

	W_0	W	P_f ratio[a]	Mean stress before optim	Mean stress after optim
Wing	100	99.25	1.309	39.77	40.07
Hor tail	20	20.75	0.257	39.77	38.32
System	120	120	0.783		

[a]P_f ratio is the ratio of the probabilities of failure of the probabilistic design and deterministic design.

Table 2 Probabilistic structural optimization of wing, horizontal tail, and vertical tail system. In optimization, only the mean stresses are changed; the COV of the stresses are fixed at a COV of 0.20. Probability of failure of the wing and the tails for deterministic design are all 1×10^{-7}.

	W_0	W	P_f ratio[a]	Mean stress before optim	Mean stress after optim
Wing	100	98.80	1.531	39.77	40.25
Hor tail	20	20.67	0.300	39.77	38.48
Ver tail	10	10.53	0.149	39.77	37.78
System	130	130	0.660		

[a]P_f ratio is the ratio of the probabilities of failure of the probabilistic design and deterministic design.

With the simple relation of stress to weight, Eq. (2), and the assumptions on the distributions of stresses and failure stresses, the probability of failure can be easily calculated by a variety of methods. One of these methods will be discussed later. Then the probabilistic design optimization is solved, assuming a zero correlation coefficient between the probabilities of failure of wing and tail. We see from Table 1 that the probabilistic design and deterministic design are very close in that probabilistic design is achieved by a small perturbation of deterministic design (moving 0.75% of wing weight to tail, see columns 2 and 3). Table 1 shows that by moving 0.75% of the wing material to the tail, the probability of failure of the wing is increased by 31% of its original value. On the other hand, the weight of the tail is increased by 3.77% and thereby its probability of failure is reduced by 74%. The overall probability of failure of the wing and tail system is reduced by 22%.

The mean stresses and the COV of the stresses in the wing and the tail before and after probabilistic optimization are also listed in Table 1. We note that the mean stress in the wing is increased (by 0.76%), whereas the mean stress in the tail is reduced (by 3.6%). That is, a higher safety factor is used for the tail than the wing, but the difference is not large. For example, if the safety factor for both structures was 1.5 for the deterministic design, it would be 1.49 for the wing and 1.55 for the tail. Because aircraft companies often use additional safety factors on top of those required by the FAA code, they can slightly reduce the additional safety factor for the wing, to achieve the probabilistic design that satisfies all the FAA requirements for a deterministic design.

A striking result in Table 1 is that the ratio of probabilities of failure of the tail and the wing is about 1:5. Recall that the ratio of the tail weight and the wing weight is also 1:5. That is, at optimum, the ratio of the probabilities of failure of the components is almost equal to the ratio of their weights. This optimum probability ratio depends on the following parameters: the target probability of failure, mean and COV of the stress, and COV of the failure stress. We checked and found that the ratio of probabilities falls between 4.5 and 6.5 for a wide range of these parameters. Appendix B provides analytical proof that the ratios of the weights and probabilities are indeed approximately the same.

Recall that for deterministic design, we assumed that the probabilities of failure of the wing and tail are the same, because the components are designed with the same safety factor. So it is worthwhile to check the historical record. We researched the historical record from the National Transportation Safety Board (NTSB) [9] and found that 18 out of 717 aircraft accidents between 1973 and 2003 were due to wing structural failure, whereas 9 of the accidents were due to tail structural failure (see Appendix C). Even though wing and tail are designed with the same nominal safety factors, the large weight differential may lead to different actual safety factors. Designers may intuitively attempt to reduce the structural weight of the heavier wing by squeezing out the weight down to the limit, whereas they may be laxer with the tail. This may happen, for example, if more approximate methods with higher safety margins of safety are used for the tail. The probabilistic design supports this incentive and indicates that the design paradigm of using higher safety factors for inexpensive components can further be exploited to increase the structural safety of aircraft.

Next, we added a vertical tail to the wing and horizontal tail system. For a typical transport aircraft, the structural weight of the vertical tail is about 10% of that of the wing. The weights of the wing, the horizontal tail, and the vertical tail of our representative system before probabilistic optimization are taken as 100, 20, and 10 units, respectively. The probability of failure of the deterministic designs of the wing and tails are taken as 1×10^{-7}. each. The results of structural optimization for safety are listed in Table 2. By moving material from the wing to the tails, the probability of failure of the wing is increased by 53%, but the probabilities of failure of the horizontal tail and the vertical tail are reduced by 70% and 85%, respectively. Table 2 demonstrates that by including the vertical tail in the system, the system probability of failure is reduced by 34%, compared with 22% with two-component (Table 1). An increase in the number of components may thus increase the safety improvement of the system.

Table 2 shows a finding similar to Table 1 in that at optimum, the ratio of the probabilities of failure of the components are nearly 10:2:1, which is the same ratio of the weights of the components. This optimum probability ratio is obtained by using different safety factors for the different components. The mean stresses and the COV of the stresses in the wing, the horizontal tail, and the vertical tail before and after probabilistic optimization are also listed in Table 2. The mean stress in the wing is increased by 1.2%, whereas the mean stress in the horizontal tail and the mean stress in the vertical tail are reduced by 3.2% and 5.0%, respectively. Again, the substantial reduction in probability of failure is accomplished with a small perturbation of the safety factor. So a company that employs an additional safety factor of just a few percent would be able to reduce it for the wing and fully comply with the FAA regulations while achieving superior safety.

III. Effect of Errors in Information About Deterministic Design

The demonstration of the payoff from probability-based design in the previous section was based on assumptions on the stress distribution and probability of failure. It is known that the calculation of probability of failure can be very sensitive to errors in distribution [7,8]. So here we seek to demonstrate that because we merely seek to obtain a design with the same weight as the deterministic design, and because the probabilistic design is close to the deterministic design, the effect of errors on the ratio of the probabilities of failure of the probabilistic design and the deterministic design is minimal. In

Table 3 Errors in the ratios of failure probabilities of the wing and tail system when the COV of the stresses is underestimated by 50%. The estimated values of the COV of the stresses for wing and tail are both 20%, whereas their actual values are both 40%. Note that the underestimate of the COV corresponds to an overestimate of the mean stress, so that its actual value is 31% percent lower than the value given in Table 1.

	Optimization based on erroneous data						True optimum		
	Optimized weight[a]	Estimated[a] P_f ratio	Actual[b] P_f ratio	% error in P_f estimate	Mean stress before optim	Mean stress after optim (assumed)	True optimal weights	True optimal P_f ratio	Mean stress after optim (true)
Wing	99.25	1.309	1.256	4.2	27.42	27.62	99.11	1.309	27.66
Hor tail	20.75	0.257	0.317	−19.0	27.42	26.42	20.89	0.257	26.24
System	120	0.783	0.786	**−0.5**				0.783	

[a]From Table 1.
[b]Note that the P_f given here is the actual P_f of the assumed optimum (obtained via an erroneous COV of the stress), which is different from the true optimum corresponding to the use of the true COV of the stress.

measuring the effects of error in the statistical data, we distinguish between loss of *accuracy* and loss of *opportunity*. That is, we report on the accuracy of our estimate of the improvement in the probability of failure compared with the deterministic design. We also report on the missed opportunity to make the design even safer, if we had the correct statistical data.

A. Errors in Coefficient of Variation of Stresses

We first assume that we underestimated the COV of the stresses in the wing and the tail by 50% and performed the optimization using the wrong COV of stresses. That is, even though the true values of the COV of the stresses for the wing and the tail are both 40%, we performed the optimization based on 20% COV and obtained the design shown in Table 1. With the overall probability of failure of the deterministic design being fixed, an underestimate of the COV must go with an overestimate of the mean. Following the procedure in Appendix A, we find that the mean is overestimated by about 45% (actual mean is 31% lower than the value used in Table 1). Table 3 shows both the error in the estimation of the probability gain and the loss of opportunity to make the design safer. For the wing, we overestimate the P_f ratio (ratio of P_{fW} of the probabilistic and the deterministic designs) by 4.2% and underestimate the P_f ratio for the tail by 19%. However, the system probability of failure is underestimated by only −0.5%, because the two errors canceled each other out (see Fig. 2). We also see from the table that the probability of failure of the true optimum is very close to our estimate of the probability of failure for the optimum obtained based on the erroneous data. This is a well-known result for the effect of a parameter on the optimum of an unconstrained problem (e.g., Haftka and Gürdal [10], Sec. 5.4). That is, the loss of accuracy is approximately equal to the opportunity loss for small changes in the design.

The variation of the component and system probability-of-failure ratios with the error in the COV of the stresses in the wing and the tail are shown in Fig. 2. We see that for negative errors (underestimated COV of stress), the P_f ratio of the wing is overestimated, whereas the P_f ratio of the tail is underestimated. The two errors mostly cancel each other out and error in the system P_f ratio (and hence the opportunity loss) is very small. Similarly, for positive errors (overestimated COV of stress), even though the P_f ratio of the wing is underestimated and the P_f ratio of the tail is overestimated, the estimate of system P_f ratio is quite accurate over a wide range of error magnitude. As important is that we lose very little in terms of the potential improvement in the probability of failure due to the error. The smallness of the opportunity loss is a manifestation of the fact that the optimum ratio of the probabilities of failure is insensitive to the coefficient of variation of the stress.

We have this remarkable insensitivity of ratio of probabilities of failure to errors, because the probabilistic design is close to the

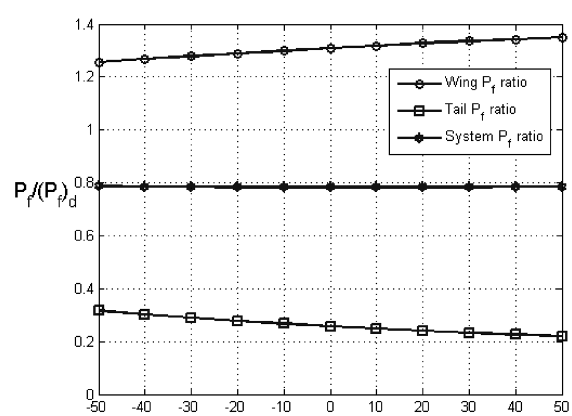

Fig. 2 The change of the ratios of probabilities of failure of the probabilistic design of Table 1 versus the error in COV (σ). Negative errors indicate an underestimate, whereas positive errors indicate an overestimate.

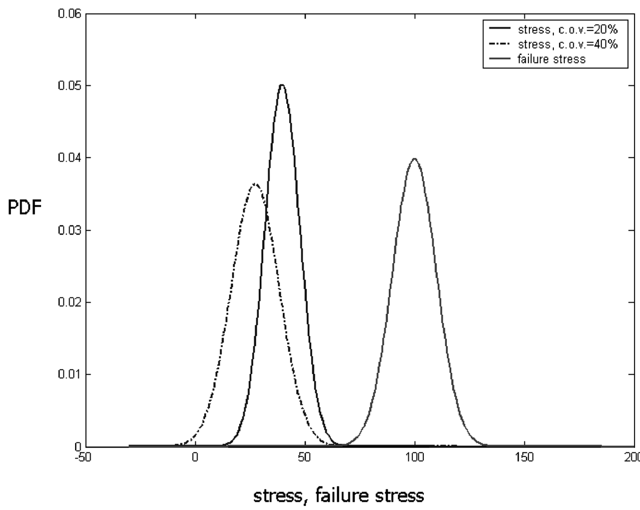

Fig. 3 Two different stress distributions at the wing leading to the same probability of failure of 1×10^{-7}.

deterministic design. For a given probability of failure of the deterministic design, errors in the mean lead to compensating errors in the standard deviation, as shown in Fig. 3, which shows two different possible distributions of the stresses in the wing (one with a COV of 20% and the other with a COV of 40%) leading to the same probability of failure, 1×10^{-7}. We observe in Fig. 3 that when the COV is 20%, the mean stress is 39.77, whereas the mean stress is lower (27.42) for a higher COV of 40%, so that they both lead to the same probability of failure. Of course, these errors can greatly affect the probability of failure. Had we performed a probabilistic design for a given probability of failure, these errors could have caused us to get a design that was much less safe than the deterministic design. To complete the investigation, Sec. III.C covers the effect of erroneous estimates of probability of failure of deterministic design.

B. Erroneous Mean Stresses

Instead of erroneous estimates for the COV of stresses, we now check the effect of errors in estimates of the mean stresses in the wing and the tail. We first assume that we underestimated the mean stresses in the wing and the tail by 20% for the wing and the tail system of Table 1. That is, even though the true values of mean stresses in the wing and the tail are both 49.71, we underestimated them as 39.77 to obtain the design of Table 1. Because the overall probability of failure of the deterministic design is fixed, an underestimate of the mean stress must go with an overestimate of the coefficient of variation. Following the procedure in Appendix A, we find that the COV is overestimated by about 193% (the actual COV is 48% lower than the value used in Table 1). Table 4 shows that underestimation of mean stresses leads to underestimating the wing P_f ratio by 4.9%, and overestimating the tail P_f ratio by 27.6%. On the other hand, the system probability-of-failure ratio is estimated with a very small error, because the two errors mostly cancel each other out. Comparing Tables 1 and 4, we see that an underestimate of the mean stresses led to an overestimate of the COV of the stresses and thus compensated for the errors in probability-of-failure estimations.

Figure 4 shows that negative errors (underestimated mean stress) lead to overestimated probability-of-failure ratio of the wing and underestimated probability-of-failure ratio of the tail. However, the two errors are mostly cancelled and the error in the system-failure-probability ratio estimation is very small. Positive errors have the opposite effect.

C. Errors in Probability-of-Failure Estimates of Deterministic Design

Mansour [11] showed that there can be a significant variation in the failure probabilities of designs constructed using the same deterministic code. Supposedly, this reflects the effect of errors in predicting structural failure that may be different between designers,

Table 4 Errors in the ratios of failure probabilities of the wing and tail system when the mean stresses are underestimated by 20%. The estimated values of the mean and the COV of the stresses for wing and tail are both (39.77, 20%). Note that the underestimate of the mean stress corresponds to an overestimate of the coefficient of variation, so that its actual value is 48% percent lower than the value given in Table 1 (COV=0.1038).

	Optimization based on erroneous data						True optimum		
	Optimized weight[a]	Estimated[a] P_f ratio	Actual[b] P_f ratio	% error in P_f estimate	Mean stress before optim	Mean stress after optim (assumed)	True optimal weights	True optimal P_f ratio	Mean stress after optim (true)
Wing	99.25	1.309	1.377	−4.9	49.71	50.09	99.36	1.309	50.03
Hor tail	20.75	0.257	0.198	29.4	49.71	47.91	20.64	0.257	48.18
System	120	0.783	0.788	**−0.6**				0.783	

[a]From Table 1.
[b]Note that the P_f given here is the actual P_f of the assumed optimum (obtained via an erroneous COV of the stress), which is different from the true optimum corresponding to the use of the true COV of the stress.

Table 5 Errors in the ratios of failure probabilities of the wing and tail system when the probability of failure of the deterministic design is underpredicted. The actual P_f is 10^{-5}, whereas it is predicted as 10^{-7}. Note that the COV of the stress is 20%.

	Optimized weight	Estimated = actual[a] P_f ratio	Mean stress before optim	Mean stress after optim (assumed)	True optimal weights	True optimal P_f ratio	% safety loss	Mean stress after optim (true)
Wing	99.25	1.240	45.91	46.26	99.05	1.310	−5.4	46.35
Hor tail	20.75	0.336	45.91	44.24	20.95	0.253	32.9	43.83
System	120	0.788				0.782	**0.8**	

[a]Estimated and actual probabilities of failure of the assumed optimum are the same, because the mean and the COV of the stress do not involve any error.

Table 6 Errors in the ratios of failure probabilities of the wing and tail system when the probability of failure of the deterministic design is underpredicted. The actual P_f is 10^{-9}, whereas it is predicted as 10^{-7}.

	Optimized weight	Estimated = actual[a] P_f ratio	Mean stress before optim	Mean stress after optim (assumed)	True optimal weights	True optimal P_f ratio	% safety loss	Mean stress after optim (true)
Wing	99.25	1.374	35.33	35.60	99.36	1.306	5.1	35.56
Hor tail	20.75	0.202	35.33	34.05	20.64	0.261	−22.4	34.24
System	120	0.788				0.783	**0.6**	

[a]Estimated and actual probabilities of failure of the assumed optimum are the same, because the mean and the COV of the stress do not involve any error.

companies, or materials. This means that the probability-of-failure estimate of the deterministic design that we used in previous calculations may be inaccurate. To address this issue, we explore the sensitivity of the ratio of the probabilities of failure of the probabilistic design and the deterministic design to erroneous estimates of probability of failure of the deterministic design. Note, however, that we still assume that the several structural components are made from the same material and are designed for the same failure mode, so that they have approximately the same probability of failure in the deterministic design.

We consider an underestimate of the probability of failure of deterministic design by two orders of magnitude. That is, we assume that we performed probabilistic design by taking the probability of failure of deterministic design as 10^{-7} instead of using the true value of 10^{-5}. Table 5 shows that underestimated P_f of deterministic design leads to transferring a lower amount of material (0.75%, column 6) than optimum (0.95%, column 5) from the wing to the tail. However, even though the wing is designed to be 5.4% safer than the true optimum and the tail is designed to be 32.9% less safe, the actual system probability-of-failure ratio is only 0.8% larger than its estimated value (columns 2–4).

Similarly, we checked what happens when we overestimate the probability of failure of the deterministic design by two orders of magnitude. Table 6 shows results similar to Table 5, but this time a larger amount of material is transferred from the wing to the tail, compared wth the true optimum (columns 5 and 6). This time, the wing is designed to be 5.1% less safe, the tail is designed to be 22%

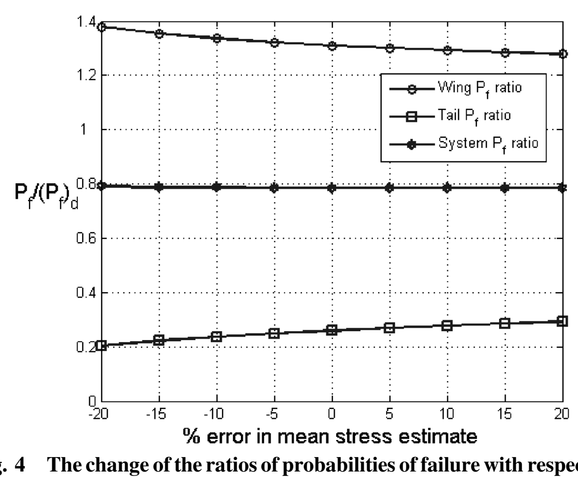

Fig. 4 The change of the ratios of probabilities of failure with respect to the error in mean stress. The negative errors indicate an underestimate, whereas the positive errors indicate an overestimate.

safer than their optimum values, and the probability-of-failure ratio is only 0.6% greater than its estimated value (columns 2–4).

D. Effect of Using Wrong Probability Distribution Type for the Stress

Apart from the parameters we investigated (COV of the stresses, the mean stresses, and the probability-of-failure deterministic design), the distribution type of the stress also affects the results of probabilistic design. Here, we explore the sensitivity of the probabilistic design to using the wrong distribution type for the stress. We assume that even though the stress follows the lognormal distribution, the optimization is performed using a normal probability distribution to obtain the results in Table 1.

Table 7 shows that if the true stress probability distribution is lognormal, the P_f ratio of the wing is overestimated and the P_f ratio of the tail is underestimated, so that a smaller amount of material is moved from the wing to the tail, compared with the true optimum design. The error in the total system P_f ratio estimate is only 2.3%. Also, as in Tables 3–6, the loss of accuracy is approximately equal to the opportunity loss, because the changes in the design are small. The loss of optimality reflects the fact that with lognormal distribution of the stress, it is advantageous to transfer more material from the wing to the tail than with the normal distribution. Of course, the true distribution may be different from lognormal, however, the insensitivity is still encouraging. It is also interesting to note that even with the lognormal distribution, the optimal probabilities of failure are still proportional to the weight of the two components. Analytically, we have obtained a proof of this phenomenon only for the normal distribution (see Appendix B).

IV. Approximate Probabilistic Design Based on Failure Stress Distributions

One of the main barriers to the application of probabilistic structural optimization is computational expense. Probabilistic structural optimization is expensive, because repeated stress calculations [typically, finite element analysis (FEA)] are required for updating probability calculation as the structure is being changed. That is, the simplified approach that we used in Eq. (2) is replaced by costly FEAs.

Traditionally, RBDO is performed based on a double-loop optimization scheme, in which the outer loop is used for design optimization, whereas the inner loop performs a suboptimization for reliability analysis using methods such as first-order reliability method (FORM). Because this traditional approach is computationally expensive, even prohibitive for problems that require complex FEA, alternative methods have been proposed by many researchers (e.g., Lee and Kwak [12], Kiureghian et al. [13], Tu et al. [14], Lee

Table 7 Errors in the ratios of failure probabilities of the wing and tail system if the optimization is performed using the wrong probability distribution type for the stress (assumed is normal, true is lognormal). The probability of failure of the deterministic design is 10^{-7}. The COV of the stress is 20%.

	Optimization assuming stress is normal						Optimization using the correct distribution type: lognormal			
	Optimized weight[a]	Estimated[a] P_f ratio	Actual[b] P_f ratio	% error in P_f estimate	Mean stress before optim	Mean stress after optim (assumed)	True optimal weights	True optimal P_f ratio	Mean stress before optim	Mean stress after optim (true)
Wing	99.25	1.309	1.201	9.0	32.04	32.28	98.90	1.307	32.04	32.39
Hor tail	20.75	0.257	0.402	−36.1	32.04	30.87	21.10	0.265	32.04	30.36
System	120	0.783	0.801	**−2.3**				0.786		

[a]From Table 1.
[b]The actual P_f of the optimum obtained via an erroneous stress distribution type.

et al. [15], Qu and Haftka [16], and Du and Chen [17]). These methods replace the probabilistic optimization with sequential deterministic optimization using inverse reliability measures to reduce the computational expense. The downside of these approaches is that they do not necessarily converge to the optimum design. We note, however, that most of the computational expense is associated with repeated stress calculation and we have just demonstrated insensitivity to the details of the stress distribution. This allows us to propose an approximate probabilistic design approach that might lead to a design nearer the optimum (depending, of course, on the accuracy of the approximation).

Structural failure, using most failure criteria, occurs when a stress σ at a point exceeds a failure stress σ_f. For a given deterministic stress σ, the probability of failure is

$$P_f = \text{prob}(\sigma \geq \sigma_f) = F(\sigma) \tag{4}$$

For random stress, the probability of failure is calculated by integrating Eq. (4) for all possible values of the stress σ

$$P_f = \int F(\sigma) s(\sigma)\, \mathrm{d}\sigma \tag{5}$$

For the calculations of the probability of failure in the preceding section, numerical integration of Eq. (5) was performed. It is clear from Eq. (5) that accurate estimation of probability of failure requires accurate assessments of the probability distributions of the stress and the failure stress. For the failure stress, the FAA requires aircraft builders to perform characterization tests, use them to construct a statistical model, and then select failure allowables (A-basis or B-basis values) based on this model. Hence, the statistical characterization of the failure stress is solid. On the other hand, the probability density function of the stress, $s(\sigma)$, is poorly known, because it depends on the accuracy of structural and aerodynamic calculations, the knowledge of the state of the structure, damage progression, and pilot actions. As we discussed earlier, it is reasonable to assume that stress is normally distributed, because a large number of sources contribute to the uncertainty in stress, such as errors in load and stress calculations, variabilities in geometry, loads, and material properties. The reader is referred to Acar et al. [2] for a more detailed discussion on the sources of uncertainty in stress.

By using the mean value theorem, Eq. (5) can be rewritten as

$$P_f = F(\sigma^*) \int s(\sigma)\, \mathrm{d}\sigma = F(\sigma^*) \tag{6}$$

where the second equality is obtained by using the fact that the integral of $s(\sigma)$ is 1. Equation (6) basically states that the effect of the poorly characterized probability of the stress can be boiled down to a single characteristic-stress value σ^*. This value can be obtained by estimating $s(\sigma)$ and integrating as specified in Eq. (6). However, it is equally possible to use historical data on probabilities of failure of aircraft structural components to do the reverse. That is, given an estimate of the probability of failure, we can obtain the characteristic stress σ^* that corresponds to this historical aircraft accident data

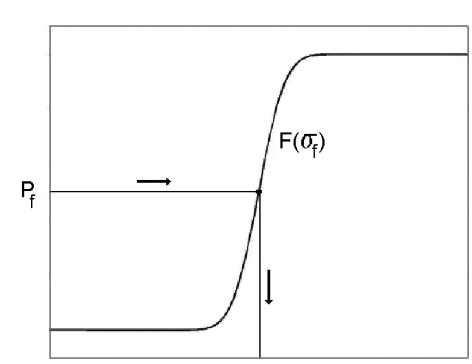

Fig. 5 Calculation of characteristic stress σ^* from probability of failure.

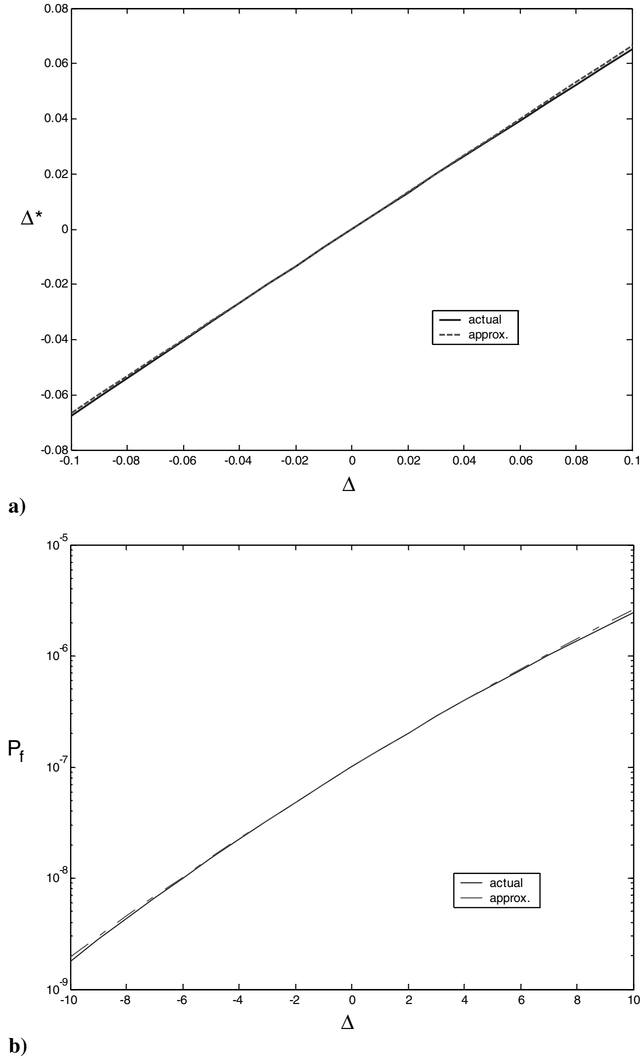

Fig. 6 Evaluating the accuracy of approximation for the characteristic stress: a) comparison of approximate and exact Δ and Δ* and b) the resulting probabilities of failure for lognormal failure stress (with a mean of 100 and a COV of 10%) and normal stress (with a mean of 39.77 and a COV of 20%).

when airplanes are designed using the deterministic FAA process (see Fig. 5).

Recall that in probabilistic design of the wing and tail system, we deviate from the deterministic process by reducing the structural margin on the wing and increase the margin on the tail, assuming that the structural redesign changes the stress distribution by simple scaling of σ to $\sigma(1+\Delta)$. Under this simple stress scaling, the characteristic stress will change from σ^* to $\sigma^*(1+\Delta^*)$ to allow probabilistic design with a minimum number of stress analyses. We assume here that the relative change in the characteristic stress is proportional to the relative change in the stress, that is,

$$\Delta^* = k\Delta \tag{7}$$

The value of k depends on the mean and COV of the stress and the failure stress. For lognormally distributed failure stress with a mean of 100 and a COV of 10%, and normally distributed stress with a mean of 39.77 and a COV of 20% (the values from our representative example), Fig. 6a shows the relation of Δ and Δ^*. Notice that the variation is almost linear. Figure 6b shows the effect of the Δ^* approximation on the probability of failure. We see that the linearity assumption is quite accurate over the range $-10\% \leq \Delta \leq 10\%$. Figure 6a that shows the variation of the change in characteristic stress with the change in stress is generated as follows. Given a failure-stress distribution (lognormal with 100 mean and 10% COV) and a stress distribution (normal with 39.77 mean and 20% COV), we have a probability of failure P_{f1} that corresponds to a characteristic stress of σ_1^* value [Eq. (6)]. When the stress distribution is shifted by Δ [that is, σ is changed to $\sigma(1+\Delta)$], the probability of failure changes to P_{f2}, which changes the characteristic stress to σ_2^*, then the ratio σ_2^*/σ_1^* is equal to $1+\Delta^*$. The stress change Δ is varied between -10 and 10%, and the corresponding characteristic-stress change is plotted.

V. Application of Characteristic-Stress Method to Wing and Tail Problem

In this section, we apply the probability-of-failure estimation to the wing and tail problem. As in Sec. II.B, the weights of the wing and the tail before probabilistic optimization are taken as 100 and 20 units, respectively. The probability of failure of the wing and the tail are both taken as 1×10^{-7}. The failure stress of the wing and tail materials is assumed to follow lognormal distribution with a mean value of 100 and 10% COV. The coefficients of variations of the stresses in the wing and the tail are assumed to be 20%. The correlation coefficient for probabilities of failure of wing and tail is assumed to be zero.

As we discussed in Sec. II, some material is taken from the wing and added to the tail so that stresses in the wing and the tail are scaled by $1+\Delta_W$ and $1+\Delta_T$, respectively. Similarly, the characteristic stresses in the wing and the tail, σ_W^* and σ_T^*, are scaled by $1+\Delta_W^*$ and $1+\Delta_T^*$, respectively. The probabilistic design optimization problem stated earlier in Eq. (3) can now be reformulated as

$$\min_{W_W, W_T} P_f^*(W_W, W_T) = [1 - P_{fW}^*(W_W)][1 - P_{fT}^*(W_T)]$$
$$\text{such that } W_W + W_T = W_{dW} + W_{dT} \tag{8}$$

The weights of the components and characteristic stresses are related via

$$\sigma_W^* = (1 + k_W \Delta_W)\sigma_{dW}^*, \qquad \sigma_T^* = (1 + k_T \Delta_T)\sigma_{dT}^* \tag{9}$$

where the relative changes in the stresses are calculated from

$$\Delta_W = \frac{W_{dW}}{W_W} - 1, \qquad \Delta_T = \frac{W_{dT}}{W_T} - 1 \tag{10}$$

The probabilistic optimization problem stated in Eq. (8) is solved, and the probabilities of failure are computed. Table 8 shows that the P_f ratios of the wing and the tail are estimated as 1.307 and 0.263, instead of their actual values of 1.305 and 0.261. So the characteristic-stress method estimates the system P_f ratio as 0.785, whereas the actual P_f ratio corresponding to the redesign is 0.783, which is the same system P_f ratio in Table 1.

Table 8 shows that the error associated with the approximation of the characteristic stress in Eq. (7) is small. This is expected, based on Fig. 6, which shows that the approximation of Δ^* is very good. However, the main issue here is to show what happens if we commit errors in evaluating the k value in Eq. (7) due to errors in the distribution parameters in the stresses and the failure stresses. We investigated the effects of overestimating and underestimating k with 20% error. Table 9 shows that a 20% underestimate of k leads to designing the wing for a higher P_f ratio (1.392 instead of 1.306) and designing the tail for a lower P_f ratio (0.187 instead of 0.269). The

Table 8 Probabilistic design optimization for safety of the representative wing and tail system using the characteristic-stress method. The COV of the stresses in the wing and tail are both 20%.

	Table 1		Proposed method		
	ΔW	P_f ratio	ΔW[a]	P_f^* ratio	Actual P_f ratio
Wing	−0.75	1.309	−0.75	1.307	1.305
Hor tail	3.77	0.257	3.73	0.263	0.261
System	0.0	0.783	0.0	0.785	0.783

[a]$\%\Delta W$ is percent changes in weight.

Table 9 Effect of 20% underestimate of *k* on the ratios of probability-of-failure estimate

	k		Table 1		Characteristic-stress method with erroneous k		
	Correct	With 20% underestimate	ΔW	P_f ratio	ΔW^{a}	P_f^* ratio	Actual P_f ratio
Wing	0.664	0.532	−0.75	1.309	−0.93	1.306	1.392
Hor tail	0.664	0.532	3.77	0.257	4.64	0.269	0.187
System			0.0	0.783	0.0	0.787	0.790

[a]$\% \Delta W$ is percent changes in weight

overall system P_f ratio, however, is increased by only 0.4%. The variation of P_f ratio of the wing, the tail, and the system with the error in k is depicted in Fig. 7. It is seen that the effect is small for a wide range of errors.

For a more complicated problem, when the stresses are calculated via FEA, the application of the proposed method is as follows. After calculating the stresses from FEA, the relative changes in the stresses (i.e., the Δ values) are calculated. Then the characteristic stresses are updated by using Eqs. (7) and (9). Finally, the probabilities of failure of the components are updated using Eq. (6). The computational expenses with probability calculations are reduced greatly, and the probabilistic optimization problem is reduced to a semideterministic optimization problem.

VI. Conclusions

Probabilistic structural design achieves better performance than deterministic design by applying higher safety factors to lower-weight components. This was demonstrated on a design problem of distributing structural material between the wing, horizontal tail, and vertical tail of a typical airliner. Although deterministic design leads to similar probabilities of failure for the three components, the probabilistic design led to probabilities of failure that were approximately proportional to the structural weight of the component. This result was shown to be a property of the normal distribution. Remarkably, even though the ratios of weights and probability-of-failure ratios of the three components were 10:2:1, this was accomplished by reducing the safety factor on the wing by only about 1% and using the material to increase the safety factor on the horizontal and vertical tails by 3 and 5%, respectively. This redistribution led to a reduction of 34% in the probability of failure for the same total weight. The small perturbation of the safety factor can probably be accommodated by the additional safety factors that aircraft companies often use on top of those required by the FAA code. So the aircraft companies can slightly reduce these additional safety factors for the wing, to achieve the probabilistic design that satisfies all of the FAA requirements for a deterministic design!

We used estimates of the probability of failure of the deterministic design (obtained from historical record) as the starting point of the probabilistic optimization. Because the exact values of the probability of failure of the deterministic design and the parameters of the probability distribution of structural response are rarely known, we checked the sensitivity of the ratio of probabilities of the probabilistic design and the deterministic design to large inaccuracies in the parameters of the stress distribution, type of distribution, and probability-of-failure estimate of the deterministic design. In particular, 50% errors in the standard deviation of the stress or 20% error in the mean stress led to less than 1% difference in the probability-of-failure ratios (i.e., ratio of $P_f s$ of the probabilistic and the deterministic designs). We also found that two orders of magnitude of error in the probability-of-failure estimate of the deterministic design led to less than 1% difference in the system probability-of-failure ratio.

Finally, these results inspired us to offer an approximate characteristic-stress method that dispenses with most of the expensive structural response calculations (typically done via finite element analysis). We showed that this approximation still leads to similar redistribution of material between structural components and similar system probability of failure.

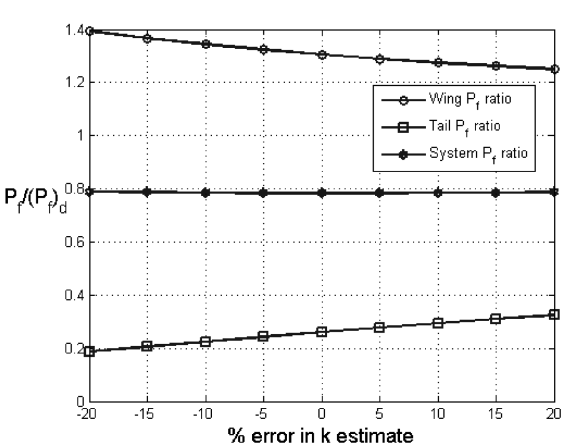

Fig. 7 The variation of the ratios of probabilities of failure with respect to the error in *k*. The negative errors indicate an underestimate, whereas the positive errors indicate an overestimate.

Appendix A: Calculating the Mean and COV of Stress Distribution Using Failure Probability Information

The probability of failure is defined in terms of the probability distribution functions of the stress and the failure stress in Eq. (5).

$$P_f = \int F(\sigma) s(\sigma)\, \mathrm{d}\sigma$$

We assume that the stress follows normal distribution. The parameters of the normally distributed stress are the mean μ_σ and standard deviation (or coefficient of variation c_σ can also be used instead of standard deviation). We assume that material characterization tests provide us with an accurate failure-stress distribution. If the failure stress also follows normal distribution with a mean value of μ_f and coefficient of variation of c_f, then Eq. (5) can be reduced to

$$P_f = 1 - \Phi(\beta) \tag{A1}$$

where Φ is the cumulative distribution function of the standard normal distribution and β is the reliability index, which is calculated as

$$\beta(\mu_\sigma, c_\sigma) = \frac{\mu_f - \mu_\sigma}{\sqrt{\mu_f^2 c_f^2 + \mu_\sigma^2 c_\sigma^2}} \tag{A2}$$

Now consider the reverse situation. Given the estimate of the probability of failure $P_{f\mathrm{given}}$ and the coefficient of variation of the stress c_σ, we can compute the mean value of the stress μ_σ from

$$\mu_\sigma = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{A3}$$

where $A = \beta_{\mathrm{given}}^2 c_\sigma^2 - 1$, $B = 2\mu_f$, $C = \mu_f^2(\beta_{\mathrm{given}}^2 c_f^2 - 1)$, and $\beta_{\mathrm{given}} = \Phi^{-1}(1 - P_{f\mathrm{given}})$.

Similarly, if the mean value of the stress is known, then the coefficient of variation of the stress can be calculated from Eq. (A1).

When the failure stress follows lognormal distribution, then the probability of failure is calculated via the integral given in Eq. (5). Hence, given the distribution parameters of lognormally distributed failure stresses λ_f and ζ_f, probability of failure is a function of the mean and coefficient of variation of the stress, as given in Eq. (A4):

$$P_f(\mu_\sigma, c_\sigma) = \int F(\lambda_f, \zeta_f, \sigma) s(\mu_\sigma, c_\sigma, \sigma)\,\mathrm{d}\sigma \tag{A4}$$

Given the estimate of the probability of failure $P_{f\text{given}}$, the mean and coefficient of variation of the stress distribution can be calculated from

$$P_f(\mu_\sigma, c_\sigma) - P_{f\text{given}} = 0 \tag{A5}$$

Equation (A5) is nonlinear in terms of μ_σ and c_σ. When c_σ is known, then μ_σ can be calculated using the bisection method or Newton's method. Alternatively, if the mean value of the stress is known, then the coefficient of variation of the stress can be calculated.

The computations are performed using MATLAB, which has the following built-in functions for numerical computations. Equation (A5) can be solved for the mean or coefficient of variation of the stress using the function `fzero`, which uses a combination of bisection, secant, and inverse quadratic interpolation methods. The integral given in Eq. (A4) can be computed using the function `quadl`, which numerically evaluates the integral using an adaptive Lobatto quadrature technique. The integrand of Eq. (A4) can be easily calculated using MATLAB. For normally distributed stress, the function `normpdf` can be used to compute the probability density function s(σ), and for lognormally distributed failure stress, the function `logncdf` can be used to compute the cumulative distribution function $F(\sigma)$.

Appendix B: Relation of Component Weights and Optimum Component Failure Probabilities

In Sec. II.B, we found that the ratio of probabilities of failure of the wing and the tail are very close to the ratio of their weights. Here, we aim to provide an analytical proof by using some approximations.

The probability of failure of the wing and tail system is defined as

$$P_f = 1 - (1 - P_{fW})(1 - P_{fT}) \tag{B1}$$

Let w be the weight transferred from the wing to the tail as a result of probabilistic optimization. The optimality condition of optimization for safety is

$$\frac{\partial P_f}{\partial w} = (1 - P_{fW})\frac{\partial P_{fT}}{\partial w} + (1 - P_{fT})\frac{\partial P_{fW}}{\partial w} \cong \frac{\partial P_{fT}}{\partial w} + \frac{\partial P_{fW}}{\partial w} = 0 \tag{B2}$$

Noting that $P_{fT} = F(\sigma_T^*)$ and $P_{fW} = F(\sigma_W^*)$ and using the chain rule, the partial derivatives in Eq. (B2) can be written as

$$\frac{\partial P_{fT}}{\partial w} = \frac{\partial P_{fT}}{\partial \sigma_T^*}\frac{\partial \sigma_T^*}{\partial w} = f_T\frac{\partial \sigma_T^*}{\partial w}, \qquad \frac{\partial P_{fW}}{\partial w} = \frac{\partial P_{fW}}{\partial \sigma_W^*}\frac{\partial \sigma_W^*}{\partial w} = f_W\frac{\partial \sigma_W^*}{\partial w} \tag{B3}$$

where and f_T and f_W are the values of probability density function of the failure stress evaluated at σ_T^* and σ_W^*, respectively. That is, $f_T = f(\sigma_T^*)$ and $f_W = f(\sigma_W^*)$. Now, combining (B2) and (B3), we get

$$f_T\frac{\partial \sigma_T^*}{\partial w} + f_W\frac{\partial \sigma_W^*}{\partial w} = 0, \qquad \frac{f_T}{f_W} = -\frac{\partial \sigma_W^*/\partial w}{\partial \sigma_T^*/\partial w} \tag{B4}$$

Recall that we assumed the stresses and weights are inversely proportional, that is,

$$\begin{aligned}\sigma_W^* &= \frac{W_{dW}}{W_W}\sigma_{dW}^* = \frac{W_{dW}}{W_{dW} - w}\sigma_{dW}^* \\ \sigma_T^* &= \frac{W_{dT}}{W_T}\sigma_{dT}^* = \frac{W_{dT}}{W_{dT} + w}\sigma_{dT}^*\end{aligned} \tag{B5a}$$

Then the ratio of partial derivatives $\partial\sigma_W^*/\partial w$ and $\partial\sigma_T^*/\partial w$ can be approximated as

$$\frac{\partial \sigma_W^*/\partial w}{\partial \sigma_T^*/\partial w} = \frac{\frac{W_{dW}}{(W_{dW}-w)^2}\sigma_{dW}^*}{-\frac{W_{dT}}{(W_{dT}+w)^2}\sigma_{dT}^*} \cong -\frac{W_{dT}}{W_{dW}} \tag{B5b}$$

where the second equality holds true (because the moved material w is much smaller than both component weights W_{dT} and W_{dW}) and the deterministic characteristic stresses are equal (because they are made of the same material and designed for the same probability of failure). Then Eqs. (B4) and (B5) can be combined to yield

$$\frac{f_T}{f_W} \cong \frac{W_{\mathrm{d}T}}{W_{\mathrm{d}W}} \tag{B6}$$

Now we need to relate the ratio of PDFs to the ratio of probabilities of failure. The probability of failure is defined in terms of the PDF of the failure stress as

$$P_f = \int_{-\infty}^{\sigma^*} f(x)\,\mathrm{d}x \tag{B7}$$

If we assume that the failure stress is normally distributed, then the probability of failure can be written as

$$P_f = \int_{-\infty}^{s^*} c e^{-x^2/2}\,\mathrm{d}x \tag{B8}$$

where c is a constant and s^* is the characteristic stress scaled with the mean and standard deviation of the failure stress, as given in Eq. (B9):

$$c = \frac{1}{\sqrt{2\pi}}, \qquad s^* = \frac{\sigma^* - \mu_f}{\mathrm{std}_f} \tag{B9}$$

where μ_f and std_f are the mean and standard deviation of the failure stress. Because the normal distribution is symmetric, the probability of failure, Eq. (B8), can be rewritten as

$$P_f = \int_{-\infty}^{s^*} c e^{-x^2/2}\,\mathrm{d}x = \int_{-s^*}^{\infty} c e^{-x^2/2}\,\mathrm{d}x \tag{B10}$$

The probability of failure can be reformulated by using the equality, Eq. (B11), given in [18] (p. 298, Eq. 7.1.14):

$$2e^{z^2}\int_z^\infty e^{-x^2}\,\mathrm{d}x = \frac{1}{z+}\frac{1/2}{z+}\frac{1}{z+}\frac{3/2}{z+}\frac{2}{z+}\cdots\ (\Re z > 0) \tag{B11}$$

where

$$\frac{1}{z+}\frac{1/2}{z+}\frac{1}{z+}\frac{3/2}{z+}\frac{2}{z+}\cdots$$

is a continued fraction [denoted as $CF(z)$] that can also be written as

$$CF(z) = \frac{1}{z+}\frac{1/2}{z+}\frac{1}{z+}\frac{3/2}{z+}\frac{2}{z+}\cdots = \cfrac{1}{z + \cfrac{1/2}{z + \cfrac{1}{z + \cfrac{3/2}{z + \cfrac{2}{z + \cdots}}}}} \tag{B12}$$

Then Eq. (B12) is rewritten as

$$\int_z^\infty e^{-x^2}\,\mathrm{d}x = \frac{CF(z)}{2}e^{-z^2} \tag{B13}$$

or

$$\int_z^\infty e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2}}CF\left(\frac{z}{\sqrt{2}}\right)e^{-z^2/2} \tag{B14}$$

Then the probability of failure, Eq. (B10), is reduced to

$$P_f = \frac{c}{\sqrt{2}}CF\left(\frac{-s^*}{\sqrt{2}}\right)e^{-(-s^*)^2/2} \tag{B15}$$

Because the probabilities of failure of aircraft structures are on the

Table C1 Aircraft accidents and probability of failure of aircraft structures. Examples of first-generation airplanes are Comet 4, 707, 720, and DC-8. Boeing 727, Trident, VC-10, and 737-100/-200 are examples of second-generation airplanes. Early widebody airplanes are 747-100/-200/-300/SP, DC-10, L-1011, and A300. Examples of current-generation airplanes are MD-80/-90, 767, 757, A310, A300-600, 737-300/-400/-500, F-70, F-100, and A320/319/321.

Aircraft generation[a]	Accident rate per million departures[a]	Total number of accidents[a]	Accidents due to structural failure[a]	Structural failure rate per departure
	A	B	C	$A \times C/B$
First	27.2	49	0	0
Second	2.8	130	2	4.31×10^{-8}
Early widebody	5.3	53	2	2.00×10^{-7}
Current	1.5	161	2	1.86×10^{-8}
Total	——	393	6	——

[a]These columns are taken from the Boeing accident report .

orders of 10^{-7}, the scaled characteristic stress is negative and its absolute value is much smaller than one. That is,

$$|s^*| \gg 1, \qquad s^* < 0 \tag{B16}$$

Based on Eq. (B16), the continued fraction in Eq. (B12) can be approximated as $CF(z) \cong \frac{1}{z}$. Then the probability of failure, Eq. (B15), becomes

$$P_f \cong -\frac{c}{s^*} e^{-(s^*)^2/2} = -\frac{f(s^*)}{s^*} \tag{B17}$$

Thus, from Eq. (B17), the probabilities of failure of the wing and the tail are

$$P_{fT} \cong -\frac{f(s^*)}{s^*} = -\frac{f_T}{\left(\frac{\sigma_T^* - \mu_f}{\mathrm{std}_f}\right)}, \qquad P_{fW} \cong -\frac{f(s_W^*)}{s_W^*} = -\frac{f_W}{\left(\frac{\sigma_W^* - \mu_f}{\mathrm{std}_f}\right)} \tag{B18}$$

which leads to

$$\frac{P_{fT}}{P_{fW}} \cong \frac{f_T}{f_W}\left(\frac{\sigma_W^* - \mu_f}{\sigma_T^* - \mu_f}\right) \tag{B19}$$

Note that the stresses at the deterministic design are close to the stresses at probabilistic design, and so $\sigma_W^*/\sigma_T^* \cong 1$. Then Eq. (B19) can be simplified to

$$\frac{P_{fT}}{P_{fW}} \cong \frac{f_T}{f_W} \tag{B20}$$

Finally, combining Eqs. (B6) and (B20), we find that the ratio of failure probabilities is approximately equal to the ratio of weights, that is,

$$\frac{P_{fT}}{P_{fW}} \cong \frac{f_T}{f_W} \cong \frac{W_{dT}}{W_{dW}} \tag{B21}$$

Appendix C: Historical Record for Aircraft Probability of Failure

Because aircraft structural design still relies on deterministic optimization, we first look at the historical record on the probability of failure of traditionally (i.e., via deterministic design) designed aircraft structures. Tong [19] performed a thorough literature review on aircraft structural risk and reliability analysis. Tong refers to the paper by Lincoln [20] that reports that the overall failure rate for all systems due to structural faults is one aircraft lost in more than ten million flight hours (i.e., $P_f = 10^{-7}$ per flight hours). The Boeing Company publishes the "Statistical Summary of Commercial Jet Aviation Accidents" each year and provides data back to 1959 [21] to indicate trends. The number of accidents that occurred between 1959 and 2001 due to structural failure, the total number of accidents, and the accident rate corresponding to different aircraft generations are listed in Table C1. Table C1 shows that failure probability per departure of second-generation airplanes is 4.31×10^{-8}, whereas the failure probabilities of early widebody airplanes and current-generation airplanes are 2.0×10^{-7} and 1.86×10^{-8}, respectively.

We researched the historical record using NTSB data on aircraft operated by U.S. carriers between 1983 and 2003. We found that the number of accidents that resulted in wing failure and tail failure were 18 and 9, respectively. This indicates that the probability of failure of the tail is about half of that of the wing.

Acknowledgments

This work has been supported in part by the NASA Constellation University Institute Program (CUIP), with Claudia Meyer as program monitor, and NASA Langley Research Center grant number NAG1-03070, with W. J. Stroud as program monitor. The authors thank Efstratios Nikolaidis of the University of Toledo for his insightful comments.

References

[1] Acar, E., Kale, A., Haftka, R. T., and Stroud, W. J., "Structurally Safety Measures for Airplanes," *Journal of Aircraft*, Vol. 43, No. 1, 2006, pp. 30–38.

[2] Acar, E., Kale, A., and Haftka, R. T., "Comparing Effectiveness of Measures that Improve Aircraft Structural Safety," *Journal of Aerospace Engineering* (to be published).

[3] Lincoln, J. W., "Method for Computation of Structural Failure Probability for an Aircraft," U.S. Dept. of Defense, , ASD-TR-80-5035, Wright-Patterson Air Force Base, OH, July 1980.

[4] Wirsching, P. H., "Literature Review on Mechanical Reliability and Probabilistic Design," Probabilistic Structural Analysis Methods for Select Space Propulsion System Components (PSAM), Vol. 3, NASA CR 189159, 1992.

[5] Anon., "Integration of Probabilistic Methods into the Design Process," Society of Automotive Engineers, Aerospace Information Rept. 5080, 1997.

[6] Long, M. W., and Narciso. J. D., "Probabilistic Design Methodology for Composite Aircraft Structures," Federal Aviation Administration Final Report, DOD/FAA/AR-99/2, June 1999.

[7] Ben Haim, Y., and Elishakoff, I., *Convex Models of Uncertainty in Applied Mechanics*, Elsevier, Amsterdam, 1990.

[8] Neal, D. M., Matthews, W. T., and Vangel, M. G., "Uncertainties in Obtaining High Reliability from Stress-Strength Models," *Proceedings of the 9th DOD/NASA/FAA Conference on Fibrous Composites in Structural Design*, Vol. 1, Department of Defence, Lake Tahoe, NV, 1992, pp. 503–521.

[9] *NTSB Aviation Accident Database* [online database], http://www.ntsb.gov/ntsb/query.asp#query_start.

[10] Haftka, R. T., and Gürdal, Z., *Elements of Structural Optimization*, 3rd ed., Kluwer Academic, Norwell, MA, 1992.

[11] Mansour, A., "An Introduction to Structural Reliability Theory," Ship Structure Committee Rept. SSC-351, Washington, D.C., Jan. 1989, p. 145.

[12] Lee, T. W., and Kwak, B. M., "A Reliability-Based Optimal Design Using Advanced First Order Second Moment Method," *Mechanics of Structures and Machines*, Vol. 15, No. 4, 1987, pp. 523–542.

[13] Kiureghian, A. D., Zhang, Y., and Li, C. C., "Inverse Reliability Problem," *Journal of Engineering Mechanics*, Vol. 120, No. 5, 1994, pp. 1154–1159.

[14] Tu, J., Choi, K. K., and Park, Y. H., "A New Study on Reliability Based Design Optimization," *Journal of Mechanical Design*, Vol. 121, No. 4, 1999, pp. 557–564.
[15] Lee, J. O., Yang, Y. S., and Ruy, W. S., "A Comparative Study on Reliability-Index and Target-Performance-Based Probabilistic Structural Design Optimization," *Computers and Structures*, Vol. 80, Nos. 3–4, 2002, pp. 257–269.
[16] Qu, X., and Haftka, R. T., "Reliability-Based Design Optimization Using Probabilistic Sufficiency Factor," *Structural and Multidisciplinary Optimization*, Vol. 27, No. 5, 2004, pp. 314–325.
[17] Du, X., and Chen, W., "Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design," *Journal of Mechanical Design*, Vol. 126, No. 2, 2004, pp. 225–233.
[18] Abramovitz, M., and Stegun, I., "Handbook of Mathematical Functions," Dover, New York, 1970.
[19] Tong, Y. C., "Literature Review on Aircraft Structural Risk and Reliability Analysis," Defence Science and Technology Organisation, Aeronautical and Maritime Research and Lab., Rept. DSTO-TR-1110, 2001.
[20] Lincoln, J. W., "Aging Aircraft Issues in the United States Air Force," *41st International SAMPE Symposium and Exhibition*, Society for Advancement of Material and Process Engineering, Anaheim, CA, 1996.
[21] "Statistical Summary of Commercial Jet Airplane Accidents: Worldwide Operations 1959-2001," The Boeing Company, Commercial Airplane Group, Seattle, WA, 2001.