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RESEARCH PAPER

A reliability index extrapolation method for separable limit states

Erdem Acar[1]

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Abstract When the limit state function (or performance function) of a structure can be written as the difference of a capacity function and a response function that are expressed in terms of independent sets of random variables (i.e., when the limit state function has a separable form), efficient simulation based techniques (e.g., Separable Monte Carlo Simulation method) can be used to predict the reliability of the structure. The accuracies of these simulation based techniques, on the other hand, diminishes as the structural reliability increases. This paper proposes a reliability index extrapolation method to predict reliability of a highly safe structure that has a separable limit state function. In this method, the standard deviations of the random variables that contribute to the capacity function are artificially inflated by using a scale parameter to obtain various (smaller) scaled reliability index values (that can be predicted accurately with small number of samples). The standard deviations of the random variables that contribute to the response function are kept unchanged in order to use the same response values in prediction of various scaled reliability indices. Then, least square regression is used to build a relationship between the standard deviation scale parameter and scaled reliability index values. Finally, an extrapolation is performed to estimate the actual (higher) reliability index. The accuracy of the proposed method is evaluated through reliability assessment of mathematical and structural mechanics example problems as well as a reliability based design optimization problem. It is found that the proposed method can provide reasonable accuracy for high reliability index estimations with only 1000 response function evaluations.

Keywords Extrapolation · High reliability · Reliability index · Separable Monte Carlo

✉ Erdem Acar
acar@etu.edu.tr

[1] Department of Mechanical Engineering, TOBB University of Economics and Technology, Söğütözü, Ankara 06560, Turkey

1 Introduction

In reliability estimation of structures, limit state function (or performance function) of a structure is used to separate the safe and the failure regions of the random variable space. The probability of failure estimation requires calculation of the multi-dimensional integral (see (1)) of the joint probability density function of all the random variables over the failure region

$$P_f = \int \cdots \int I[g(\mathbf{x}) \leq 0] f_{\mathbf{X}}(\mathbf{x}) \mathbf{dx} \quad (1)$$

where I is the indicator function that takes the value of 1 when the condition is true and takes the value of 0 when the condition is false, $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint probability density function of the set of random variables $\mathbf{X}$ and $g(\mathbf{x})$ is the limit state function. The analytical integration of this multi-dimensional function is not possible for most real life structural problems, therefore approximate analytical and simulation based approaches have been proposed for probability of failure estimation.

The approximate analytical approaches are usually computationally inexpensive compared to simulation based approaches. The most popular analytical methods are the first order reliability method (FORM (Hasofer and Lind 1974; Rackwitz and Fiessler 1978)) and second order reliability method (SORM (Breitung 1984; Tvedt 1990)), which are based on the first order and second order expansions of the limit state function at the most probable failure point (MPP). Even though the analytical approaches are computationally advantageous, they are often suitable for mildly nonlinear limit state functions and they usually become very complicated or inefficient for real life problems (e.g., for problems with multiple failure modes).

The simulation based approaches, on the other hand, have the advantage of handling any type of limit state function and handling complicated real life problems well. The most popular simulation based approach is the Monte Carlo simulation (MCS (Rubinstein 1981)) method. Given a limited computational budget for limit state function evaluations (usually requires performing computationally expensive structural analyses, such as finite element analysis), MCS becomes inaccurate for estimating low failure probabilities. Variance reduction techniques such as importance sampling (Melchers 1989), adaptive importance sampling (Wu 1994) can be used to improve the accuracy of failure probability estimations. These methods also rely on the concept of MPP search and most MPP search algorithms may fail or give erroneous results when the limit state function is highly nonlinear or discontinuous. In such cases, simulation based methods that do not rely on MPP search such as stratified sampling (Iman and Conover 1980), subset simulation (Au and Beck 2001) or line sampling (Koutsourelakis et al. 2004) can be used. Other alternatives include utilization of metamodels (Kaymaz 2005; Gondal and Lee 2012) and tail modeling (Ramu et al. 2010; Acar 2011).

In structural mechanics problems, the limit state function of a structure can often be expressed as the response exceeding capacity. In a more general case, the limit state function of a structure can be formulated as the difference between a capacity function and a response function that are expressed in terms of independent sets of random variables (i.e., the limit state function has a separable form). In such a case, Separable Monte Carlo simulation (SMCS) method can be efficiently used (Smarslok et al. 2010; Ravishankar et al. 2010). SMCS has a beneficial property of allowing the use of different sample sizes for the response function and the capacity function. In structural mechanical problems, the response evaluations are typically performed using computationally expensive finite element analysis, whereas the capacity evaluations are computationally inexpensive. Therefore, by allowing the use of a smaller sample size for the response function and a larger sample size for the capacity function, SMCS can provide an efficient approach to predict the reliability of the structure. Given a limited computational budget for limit state function evaluations, on the other hand, the accuracy improvement gained through SMCS can be inadequate for highly safe structures due to limited sampling.

In this paper, a reliability index extrapolation method is proposed to predict reliability of a highly safe structure that has a separable limit state function. In this method, the standard deviations of the random variables that contribute to the capacity function are progressively inflated to obtain various (smaller) scaled reliability indices that can be predicted accurately with small number of samples. The standard deviations of the random variables that contribute to the response function are kept unchanged in order to use the same response function values in prediction of various scaled reliability indices. Then, least square regression is used to build a relationship between the standard deviation inflation parameter and scaled reliability index values. Finally, an extrapolation is performed to estimate the actual reliability index.

The paper is organized as follows. The existing methods are briefly described in the next section. The proposed reliability index extrapolation method is presented in Section 3. The application of the proposed method to an illustrative example is presented in Section 4. The results obtained from mathematical and structural mechanics example problems are also presented and discussed in Section 4. Finally, the summary of important conclusions are listed in Section 5.

2 The existing methods

2.1 Extrapolation for general limit states

The main premise of reliability index extrapolation can be stated as follows. First, the standard deviation of the random variables artificially inflated by using a scale parameter to obtain smaller reliability indices so called "scaled" reliability indices. Then, a functional relationship is built between the scale parameter and the scaled reliability index. Finally, the actual reliability index is predicted by using the functional relationship established. It should be noted that the scaled reliability indices can be obtained with substantially smaller computational cost.

Motivated from the asymptotic behavior of the reliability index with respect to standard deviation of the random variables in the independent and identically distributed (i.i.d.) Gaussian space, Bucher (2009) proposed an asymptotic reliability index extrapolation method. Bucher first considered the case of a linear limit state function, and suggested that this problem can be reduced to a single variable that has a standard deviation of σ by an appropriate coordinate transformation. Then, the reliability index can be formulated as

$$\beta(k) = \frac{\beta_k}{k} \tag{2}$$

where k is the scale factor and β_k is the scaled reliability index computed for the scaled standard deviation of the random variable $\sigma_k = \frac{\sigma}{k}$. The actual reliability index is equal to $\beta_{act}=\beta(k=1)$.

Bucher then considered a hyper circular limit state function in n-dimensional Gaussian space in which failure is given by $g(\mathbf{X})=R^2-\mathbf{X}^\mathrm{T}\mathbf{X}\leq 0$, and obtained the relationship between the

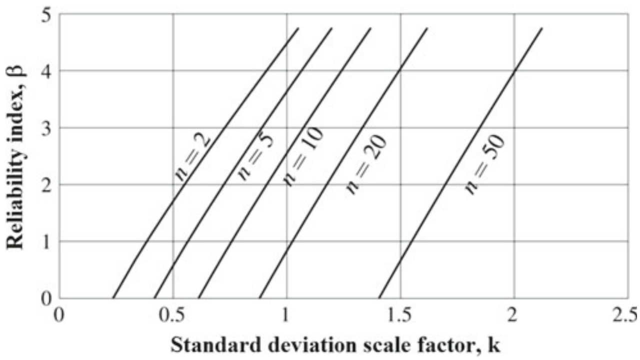


Fig. 1 Relationship between the reliability index and the standard deviation scale parameter k for hyper circular limit state function (Bucher 2009)

reliability index and the standard deviation scale parameter k in terms of the χ^2 distribution with n degrees of freedom as given below

$$\beta(k) = \Phi^{-1}\left[1-\chi^2\left(k^2R^2, n\right)\right] \tag{3}$$

The graphical depiction of the relationship between the reliability index and the standard deviation scale parameter k is shown in Fig. 1.

Based on the asymptotically linear behavior of the reliability index with respect to the standard deviation scale parameter, Bucher (2009) assumed the following functional dependence between the reliability index and the standard deviation scale parameter k

$$\beta(k) = Ak + \frac{B}{k} \tag{4}$$

Notice that as $k\to\infty$ (that is, as $\sigma_k\to 0$) the reliability index $\beta\to\infty$ so that the asymptotic behavior is ensured. To determine the coefficients A and B in (4), a set of so called "support points" $[k_i, \beta(k_i)]$ that consists of a set of scaled reliability index values obtained for a set of scale parameters $k_i<1$ are first generated, and then least square regression is performed. The number of support points affects the extrapolation results. Bucher (2009) used five support points and Sichani et al. (2011) suggested determination of the number of support points through optimization to keep the computational cost manageable.

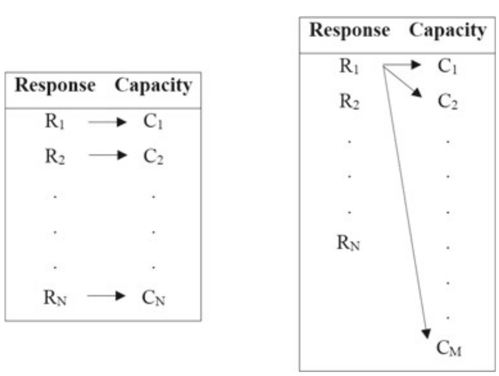


Fig. 2 Comparison of CMCS (*left*) and SMCS (*right*) sampling procedures (Ravishankar et al. 2010). In SMCS, all possible combinations of M samples of the capacity and N samples of the response are considered

2.2 Separable Monte Carlo simulation (SMCS) method

The classical Monte Carlo simulation (CMCS) estimation of the probability of failure defined in (1) is obtained from

$$\left(\hat{P}_f\right)_{CMCS} \approx \frac{1}{N}\sum_{i=1}^{N} I[g(X_i)\le 0] \tag{5}$$

where N is the number of limit state function evaluations performed.

In structural mechanics problems, the limit state function of a structure can be formulated as the difference of a capacity function and a response function that are expressed in terms of independent sets of random variables

$$g(\mathbf{X}) = g(\mathbf{X_C}, \mathbf{X_R}) = C(\mathbf{X_C}) - R(\mathbf{X_R}) \tag{6}$$

where $\mathbf{X_C}$ is the set of random variables that contribute to the capacity, and where $\mathbf{X_R}$ is the set of random variables that contribute to the response. SMCS has a beneficial property of allowing the use of different sample sizes for the response function and the capacity function (see Fig. 2). By considering all possible combinations of M samples of the capacity and N samples of the response, the SMCS estimation of the failure probability defined in (1) is obtained from

$$\left(\hat{P}_f\right)_{SMCS} \approx \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N} I\left[g\left(X_{C_i}, X_{R_j}\right)\le 0\right] \tag{7}$$

As noted earlier, the response evaluations are typically performed using computationally expensive finite element analysis, whereas the capacity evaluations are often computationally inexpensive. Therefore, by allowing the use of a smaller sample size for the response function and a larger sample size for the capacity function, SMCS can provide an efficient approach to predict the reliability of the structure. Given a limited computational budget for response evaluations (a typical value is

Table 1 Statistical properties of the random variables in the illustrative example

Random variable	Distribution	Mean	Standard deviation
X_1	Normal	8.5	1
X_2	Normal	1	0.1
X_3	Normal	1	0.1
X_4	Normal	1	0.1
X_5	Normal	1	0.1

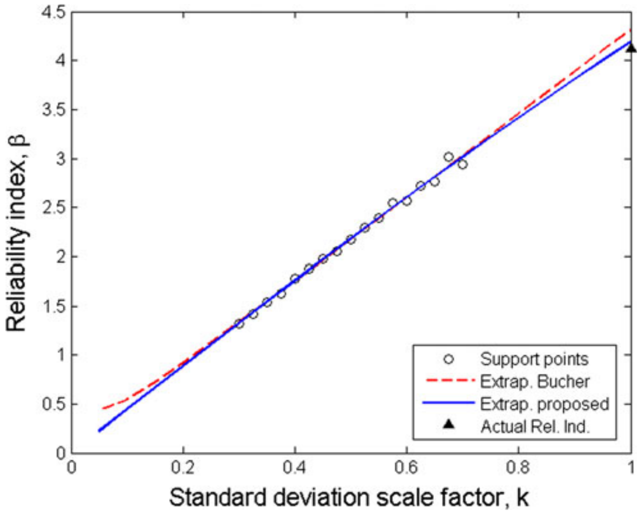


Fig. 3 Support points and extrapolation models generated for the illustrative example

1000), the accuracy improvement gained through SMCS could be inadequate for highly safe structures. In this paper, a reliability index extrapolation method is proposed to improve the accuracy of SMCS predictions as explained in the next section.

3 The proposed approach: extrapolation for separable limit states

The main computational downside of the extrapolation method presented in Section 2.1 is the necessity of performing computationally expensive response function calculations needed for evaluation of several scaled reliability indices. If the limit state function is separable, on the other hand, it is proposed in this paper that the computational cost can be alleviated by keeping the standard deviations of the random variables that contribute to the response function unchanged but scaling only the standard deviations of the random variables that contribute to the capacity function.

Consider the simple separable limit state case where the capacity function is represented with a random variable C and the response function is represented with a random variable R. Then, the limit state function can be simply written as $g(C,R)=C-R$. The computational cost of generating a realization of R is often substantially larger than generating a realization of C, so it is proposed in this paper to use an extrapolation approach in which only the standard deviation of C is scaled, whereas the standard deviation of R is fixed. Therefore, in this extrapolation approach, computation of several scaled reliability index values requires generating multiple sets of realizations of C but only a single set of realizations of R.

For the simple separable limit state case considered, assume that both C and R follows normal distribution. Then the scaled reliability index can be formulated as

$$\beta(k)=\frac{\mu_C-\mu_R}{\sqrt{\sigma_{Ck}^2+\sigma_R^2}};\quad \sigma_{Ck}=\frac{\sigma_C}{k} \tag{8}$$

Notice that as $k\to\infty$ (that is, as $\sigma_{Ck}\to 0$) the reliability index β does not show an asymptotic behavior but it has a finite value. Therefore, the extrapolation formula given in (4) is not valid. Inspired from the scaled reliability index formula given (8), this paper proposes a general formula that could be used for distribution types other than normal distribution provided that the distributions of interest are not described by higher moments. The following functional dependence between the reliability index and the standard deviation scale parameter k is proposed in this paper

$$\beta(k)=A\sqrt{\frac{1}{B/k^2+C}} \tag{9}$$

Notice that (9) satisfies the expectation that as $k\to\infty$ (that is, as $\sigma_{Ck}\to 0$) the reliability index β has a finite value.

As noted earlier, the number of support points affects the extrapolation results. Bucher (2009) used five support points and Sichani et al. (2011) suggested determination of the number of support points through optimization to keep the computational cost manageable. For the extrapolation approach proposed in this paper, the number of support points is not much an issue since only a single set of realizations of the response function is used for all support points. The location of support points also have a substantial effect on the extrapolation results. For a specific problem, on one hand it is desirable to have the largest value of the scale parameter close to one, and on the other hand the accuracy of the corresponding reliability index estimate has to be acceptable (30 % coefficient of variation of the reliability index estimate for the largest scale parameter is a

Table 2 Comparison of the reliability index predictions for the illustrative example

Method	NORFC[a]	Rel. index	Rel. index range	% Difference compared to MCS
CMCS	10^9	4.118	–	–
Extrap. Bucher	1000	4.303 (0.038)	[4.183, 4.459]	4.5
Extrap. proposed	1000	4.151 (0.110)	[3.856, 4.763]	0.8
SMCS	1000	4.212 (0.240)	[3.558, 5.062]	2.3

The numbers in the parenthesis shows the standard deviations computed over 1000 runs

[a] NORFC: number of response function calculations

Table 3 Effect of the reliability level on the performance of the proposed extrapolation method for the illustrative example

$\overline{X}_1$	CMCS	Reliability index predictions of different methods			% difference compared to CMCS predictions		
		Extrap. Bucher	Extrap. Proposed	SMC	Extrap. Bucher	Extrap. Proposed	SMC
7.5	3.199	3.335 (0.023)	3.209 (0.056)	3.209 (0.065)	4.3	0.3	0.3
8	3.659	3.818 (0.030)	3.674 (0.099)	3.686 (0.123)	4.3	0.4	0.7
8.5	4.118	4.303 (0.038)	4.151 (0.110)	4.212 (0.240)	4.5	0.8	2.3
9	4.576	4.796 (0.058)	4.650 (0.180)	N/A[a]	4.8	1.6	N/A[a]
9.5	5.007	5.298 (0.087)	5.194 (0.297)	N/A[a]	5.8	3.7	N/A[a]

The numbers in the parenthesis shows the standard deviations computed over 1000 runs

[a] mean value and standard deviation over 1000 repetitions cannot be computed because of the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels

reasonable value). This approach could also be used to determine the largest value of the scale parameter. In addition, if the support points are clustered towards zero or one, then the least square regression would put more importance towards very small or very large reliability levels, respectively, and the extrapolation results would be erroneous. In this study, the support points are generated such that the standard deviation scale factor k is changed between 0.3 and 0.7 with an interval of 0.025 based on our experience.

As the reliability index values at the support points are computed through SMCS, these reliability index values are subject to variations due to limited sampling. One approach to protect against this variation is to perform conservative reliability estimates by using bootstrap method as suggested by Picheny et al. 2010. This approach requires calculation of the reliability indices multiple times through re-sampling using the existing response and capacity function values, generating a bootstrap distribution for the reliability index, and finally performing conservative reliability estimations using the bootstrap distribution. The reader may refer to Picheny et al. (2010) for details of this approach.

Even though the proposed reliability index extrapolation approach is easy to follow and implement, it has some restrictions listed below:

- Since the proposed approach is based on scaling of the standard deviations of the capacity, the reliability estimation accuracy will reduce as the standard deviation of the capacity reduces. For the extreme case of deterministic capacity, the reliability estimations will be as accurate as the classical MCS.
- If the estimation of the capacity is expensive (e.g., when the material properties are obtained through expensive tests that need to be conducted many times), the proposed approach will lose its computational advantage.
- The proposed approach is based on the assumption that the reliability index increases monotonically as the standard deviation scale factor increases. For problems with multiple failure modes and multiple MPPs, this assumption may not always holds true. For these kinds of problems, the proposed approach is not applicable.

4 Example problems

This section provides an illustrative problem followed by an additional mathematical example problem, two structural mechanics example problems and a reliability based design optimization (RBDO) problem to compare the performance of the extrapolation formula proposed in this paper to the Bucher's extrapolation formula and the SMCS without extrapolation. Reliability index predictions of classical MCS with 10^9 samples are used as basis in comparison.

The coefficients in Bucher's extrapolation formula and the proposed extrapolation formula are estimated using the same set of support points, because the main motivation of the paper is to show that if we choose to scale only the capacity standard deviation then Bucher's extrapolation formula is not applicable and we need to use a different extrapolation formula. If the main motivation of the paper was to compare Bucher's approach to the proposed approach, then the response and the capacity standard deviations had to be scaled simultaneously and only five support points should be used for Bucher's approach. In addition, to maintain the same computational cost for both approaches, the number of response function evaluations in Bucher's approach should be five times smaller than the proposed approach.

Table 4 Statistical properties of the random variables in the additional mathematical example problem

Random variable	Distribution	Mean	Standard deviation
X_1	Lognormal	$\overline{X}_1$	10
X_2	Normal	10	0.5
X_3	Lognormal	2	0.2
X_4	Normal	2	0.3

Table 5 Evaluation of the performance of the proposed extrapolation method at various reliability levels for the additional mathematical example problem

$\overline{X}_1$	CMCS	Reliability index predictions of different methods			% difference compared to CMCS predictions		
		Extrap. Bucher	Extrap. Proposed	SMCS	Extrap. Bucher	Extrap. Proposed	SMCS
80	3.274	3.908 (0.060)	3.318 (0.075)	3.284 (0.066)	19.4	1.3	0.3
85	3.655	4.394 (0.076)	3.685 (0.096)	3.686 (0.098)	20.2	0.8	0.8
90	4.023	4.870 (0.097)	4.046 (0.123)	4.111 (0.142)	21.1	0.6	2.2
95	4.388	5.342 (0.132)	4.409 (0.167)	N/A[a]	21.7	0.5	N/A[a]
100	4.750	5.816 (0.181)	4.780 (0.227)	N/A[a]	22.4	0.6	N/A[a]
105	5.065	6.298 (0.242)	5.168 (0.308)	N/A[a]	24.3	2.0	N/A[a]

The numbers in the parenthesis shows the standard deviations computed over 1000 runs

[a] mean value and standard deviation over 1000 repetitions cannot be computed because of the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels

However, this comparative study is beyond the scope of this paper.

4.1 Illustrative example

To illustrate the proposed extrapolation approach, consider a separable limit state function $g=X_1-X_2^2-X_3^2-X_4^2-X_5^2$ in which all random variables are normally distributed with mean and standard deviation values given in Table 1. The limit state function can be rewritten as $g=C(X_1)-R(X_2,X_3,X_4,X_5)$, where the capacity function is $C(X_1)=X_1$ and the response function is $R(X_2,X_3,X_4,X_5)=X_2^2+X_3^2+X_4^2+X_5^2$. The reliability index for this problem is computed as 4.118 through classical MCS with 10^9 samples.

To construct the extrapolation models, the support points are generated first. The standard deviation scale factor k is changed between 0.3 and 0.7 with an interval of 0.025, thereby 17 support points are generated as shown in Fig. 3. The reliability indices of the support points are computed using SMCS method, where the number of response function calculation is limited to $N=1000$ and the number of capacity function calculation is limited to $M=10{,}000$. Note that for each value of the scale factor k the same response function realizations are used to reduce the computational cost, whereas different capacity function realizations are used to reduce the bias in reliability index predictions.

Since the SMCS method is a sampling based method, the reliability indices of the support points may differ from a particular set of sampling points to another. To reduce the effect of random sampling, the whole process is repeated for 1000 times and the mean values, the standard deviations and ranges of the reliability index predictions are reported in Table 2. It is seen that the prediction error of Bucher's extrapolation formula is five to six times larger than that of the extrapolation formula proposed in this paper. It is also observed that the proposed extrapolation formula has a better prediction capability compared to the SMCS without extrapolation.

Next, the effect of the reliability level on the performance of the proposed extrapolation method is evaluated. The mean value of X_1 is adjusted to attain different reliability levels for this example problem. For instance, if the mean value of X_1 is taken as $\overline{X}_1=7.5$ the corresponding reliability index is $\beta=3.199$ (computed through classical MCS with 10^9 samples), whereas if the mean value of X_1 is taken as $\overline{X}_1=9.5$ the corresponding reliability index is $\beta=5.007$ (computed through classical MCS with 10^9 samples). Table 3 shows for various reliability levels that the extrapolation formula proposed in this paper has better prediction capability compared to both the Bucher's extrapolation formula and the SMCS without extrapolation. On the other hand, the standard deviations of the reliability indices obtained from Bucher's extrapolation formula is smaller than the

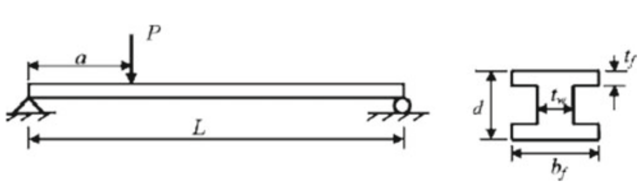


Fig. 4 The cross section and loading for the simply supported beam

Table 6 Statistical properties of the random variables for the simply supported beam

Random variable	Distribution	Mean	Standard deviation
P	Normal	6070	200
L	Normal	120	6
a	Normal	72	6
S	Normal	$\overline{S}$	$0.15\,\overline{S}$
d	Normal	2.3	1/24
b_f	Normal	2.3	1/24
t_w	Normal	0.16	1/48
t_f	Normal	0.26	1/48

Table 7 Evaluation of the performance of the proposed extrapolation method at various reliability levels for the simply supported beam

$\overline{S}$	CMCS	Reliability index predictions of different methods			% difference compared to CMCS predictions		
		Extrap. Bucher	Extrap. Proposed	SMCS	Extrap. Bucher	Extrap. Proposed	SMCS
300,000	3.112	3.222 (0.022)	3.114 (0.057)	3.119 (0.064)	3.5	0.1	0.2
350,000	3.617	3.717 (0.026)	3.627 (0.079)	3.652 (0.146)	2.8	0.3	1.0
400,000	4.004	4.092 (0.035)	4.021 (0.110)	4.147 (0.313)	2.2	0.4	3.6
450,000	4.306	4.386 (0.044)	4.342 (0.149)	N/A[a]	1.9	0.8	N/A[a]
500,000	4.547	4.629 (0.053)	4.590 (0.175)	N/A[a]	1.8	0.9	N/A[a]
550,000	4.741	4.821 (0.068)	4.817 (0.222)	N/A[a]	1.7	1.6	N/A[a]

The numbers in the parenthesis shows the standard deviations computed over 1000 runs

[a] mean value and standard deviation over 1000 repetitions cannot be computed because of the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels

extrapolation formula proposed in this paper, which is smaller than the SMCS without extrapolation. Table 3 also shows that the average performance of the SMCS cannot be assessed due to the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels. Conversely, the error of the proposed extrapolation formula in prediction of high reliability levels (a reliability index value around five) is smaller than 4 %.

4.2 Additional mathematical example problem

Consider a separable limit state function $g = \frac{X_1}{X_2} - X_3X_4$ in which X_1 and X_3 are lognormally distributed, X_2 and X_4 are normally distributed with mean and standard deviation values given in Table 4. The mean value of X_1 can be changed to attain different reliability levels, and the performance of the proposed extrapolation method can be evaluated for different reliability levels.

Table 5 presents the comparison of the average reliability index predictions of the extrapolation formula proposed in this paper to Bucher's extrapolation formula and SMCS without extrapolation. It is observed from Table 5 that Bucher's extrapolation formula leads to very large errors in reliability index predictions, even for relatively smaller reliability levels. On the other hand, the standard deviations of the reliability indices obtained from Bucher's extrapolation formula is smaller than the extrapolation formula proposed in this paper. The extrapolation formula proposed in this paper outperforms SMCS without extrapolation for all reliability levels except the smallest one considered for this example (β=3.274). Table 5 also shows that the average performance of the SMCS cannot be assessed due to the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels. Conversely, the error of the proposed extrapolation formula in prediction of high reliability levels (a reliability index value around five) is smaller than or equal to 2 %.

4.3 Simply supported beam problem

The first structural mechanics example is a simply-supported I-beam shown in Fig. 4. The beam is subjected to a concentrated load as discussed in Huang and Du (2006). This problem has a separable limit state function, which is defined as the difference between the strength (S) and the maximum normal stress (σ_{max}) due to bending as given in (10).

$$g = S - \sigma_{max} \tag{10}$$

where

$$\sigma_{max} = \frac{Pa(L-a)d}{2LI}; \quad I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12} \tag{11}$$

The statistical properties of the random variables in this example problem are given in Table 6. The mean value of

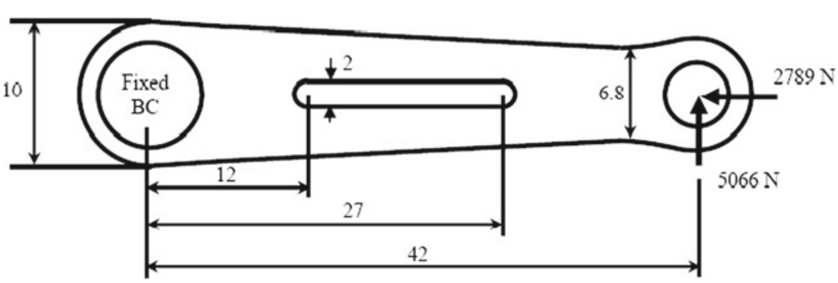


Fig. 5 The loading and boundary conditions for the torque arm problem. Dimensions are in cm (Picheny et al. 2008)

the strength ($\overline{S}$) can be changed to attain different reliability levels, and the performance of the proposed extrapolation method can be evaluated for different reliability levels. For instance, if $\overline{S}$ is taken as $\overline{S} = 300,000$ the corresponding reliability index is β=3.112, whereas if $\overline{S}$ is taken as $\overline{S} = 550,000$ the corresponding reliability index is β=4.741.

Table 7 presents the comparison of the average reliability index predictions of the extrapolation formula proposed in this paper to Bucher's extrapolation formula and SMCS without extrapolation. Table 7 shows that the extrapolation formula proposed in this paper outperforms both Bucher's extrapolation formula and SMCS without extrapolation for all reliability levels considered. However, the standard deviations of the reliability indices obtained from Bucher's extrapolation formula is smaller than the extrapolation formula proposed in this paper, which is smaller than the SMCS without extrapolation. Table 7 also shows that the average performance of the SMCS cannot be assessed due to the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels. Conversely, the error of the proposed extrapolation formula in prediction of high reliability levels (a reliability index value around five) is smaller than 2 %.

4.4 Torque arm problem

In this example problem, design of an automobile torque arm introduced by (Botkin 1982) is considered. This example problem is studied by many researchers including Kim et al. (2006) and Picheny et al. (2008). The torque arm is subjected to a horizontal load (F_x=−2789N) and a vertical load (F_y= 5066N) transmitted from a shaft at the right hole as shown in Fig. 5. The torque arm is connected to the chassis at the left hole, so fixed boundary conditions are applied at the left hole. The modulus of elasticity is equal to E=206.8 GPa, and Poisson's ratio is equal to ν=0.29 for the torque arm material. Seven design variables (d_1 through d_7) alter the shape of the torque arm as shown in Fig. 6.

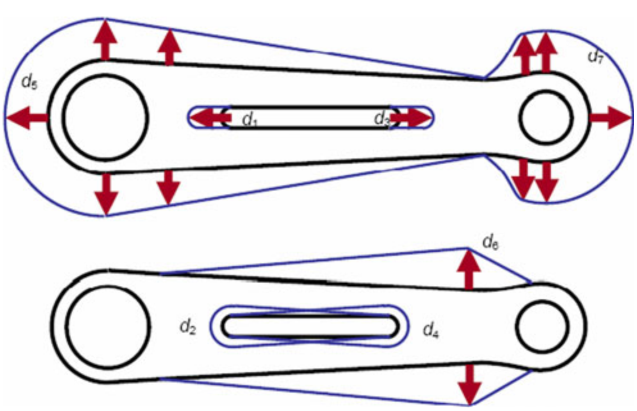


Fig. 6 Design variables for the torque arm problem (Picheny et al. 2008)

Table 8 Statistical properties of the random variables in the torque arm problem

Random variable	Distribution	Mean	Standard deviation
d_1 through d_7 (cm)	Normal	0	1
F_x (N)	Normal	−2789	278.9
F_y (N)	Normal	5066	506.6
S (MPa)	Lognormal	$\overline{S}$	0.10 $\overline{S}$

The limit state function for the torque arm problem is in separable form as given in (12).

$$g = S - \sigma_{\max} \tag{12}$$

where S is the failure stress of the torque arm material and $\sigma_{\max}$ is the maximum von Mises stress developed at the torque arm. The geometric design variables (d_1 through d_7), the applied loads (F_x and F_y) and the failure stress are taken as random variables. The statistical properties of the random variables are given in Table 8.

Since the structural analysis of the torque arm problem is complex, it is not easy to formulate a simple functional relationship between the geometry/loading parameters and the stresses in the torque arm. The maximum von Mises stress developed at the torque arm is computed through finite element analysis by using a MATLAB finite element toolbox developed by Maute (2009) and CALFEM (1999). The von Mises stress distribution in the torque arm when the design variables and the applied loads take their mean values is presented in Fig. 7. The mesh density is kept at a low level to allow for repeated analysis required for reliability analyses.

Table 9 presents the comparison of the average reliability index predictions of the extrapolation formula proposed in this paper to the Bucher's extrapolation formula and SMCS without extrapolation. Table 9 shows that Bucher's extrapolation formula leads to large errors in reliability index predictions, even for relatively smaller reliability levels. The extrapolation formula proposed in this paper outperforms SMCS without extrapolation for relatively large reliability levels, whereas SMCS without extrapolation outperforms the proposed extrapolation formula for relatively small reliability levels. In addition, the standard deviations of the reliability indices

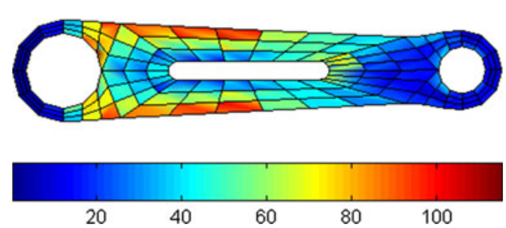


Fig. 7 The von Mises stress distribution in the torque arm when the design variables and the applied loads take their mean values. *Stresses are in MPa*

Table 9 Evaluation of the performance of the proposed extrapolation method at various reliability levels for the torque arm problem

$\overline{S}$	CMCS	Reliability index predictions of different methods			% difference compared to CMCS predictions		
		Extrap. Bucher	Extrap. Proposed	SMCS	Extrap. Bucher	Extrap. Proposed	SMCS
160	2.976	3.263 (0.038)	3.024 (0.047)	2.981 (0.055)	9.6	1.6	0.2
170	3.439	3.759 (0.044)	3.477 (0.056)	3.449 (0.075)	9.3	1.1	0.3
180	3.879	4.238 (0.054)	3.921 (0.075)	3.899 (0.106)	9.3	1.1	0.5
190	4.299	4.687 (0.067)	4.347 (0.097)	4.349 (0.167)	9.0	1.1	1.2
200	4.701	5.134 (0.091)	4.764 (0.143)	N/A[a]	9.2	1.3	N/A[a]
210	5.087	5.601 (0.122)	5.236 (0.205)	N/A[a]	10.1	2.9	N/A[a]

The numbers in the parenthesis shows the standard deviations computed over 1000 runs

[a] mean value and standard deviation over 1000 repetitions cannot be computed because of the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels

obtained from Bucher's extrapolation formula is smaller than the extrapolation formula proposed in this paper, which is smaller than the SMCS without extrapolation. Table 9 also shows that the average performance of the SMCS cannot be assessed due to the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels. Conversely, the error of the proposed extrapolation formula for predicting high reliabilities (a reliability index value around five) is smaller than 3 %.

4.5 Reliability based design optimization (RBDO) of a cantilever beam

This well-known cantilever beam design problem was first introduced by Wu et al. (2001) and investigated in different studies (Ramu et al. 2004; Qu and Haftka 2004; Ba-abbad et al. 2006). The cantilever beam depicted in Fig. 8 is subjected to two random loads F_X and F_Y. The beam may fail due to stress failure and excessive displacement. To simplify the problem, only the stress failure mode is considered in this paper. The limit state function for the stress failure mode can be written as

$$g = S - \left(\frac{6L}{wt^2} F_Y + \frac{6L}{w^2 t} F_X\right) \tag{13}$$

The applied loads (F_X and F_Y) and the yield strength S are taken as random variables and their statistical properties are given in Table 10. The minimum weight design is sought by varying the width (w) and thickness (t) of the beam, which are considered to be deterministic variables. The length of the beam is also assumed to be deterministic and taken as L= 100 in.

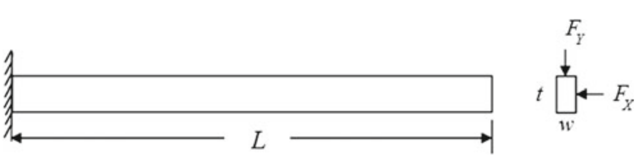


Fig. 8 The geometry and loading for the cantilever beam

All random variables in this problem follows normal distribution and the limit state function is linear with respect to the random variables, therefore the reliability of the beam can be computed easily by analytical means. Reliability based design optimization (RBDO) of the beam can be achieved by minimizing the weight of the beam such that the reliability index is larger than or equal to a target value. In this problem, the target reliability index is taken as 3.0. In this study, the RBDO problem is solved by using "fmincon" optimizer function of MATLAB is used. Since fmincon is a gradient-based optimizer, a multiple starting point strategy is used to increase the probability of converging to a global optimum solution. The optimum values of the beam width and thickness are found as w*=2.446 in and t*=3.892 in. respectively, as reported by Ramu et al. (2004).

In this paper, RBDO of the beam is also achieved by utilizing the proposed method for reliability assessment. Since the proposed method is a sampling based method, the reliability predictions would be noisy due to limited sampling and convergence problems can be faced. To eliminate this problem, metamodel based optimization approach is followed. Response surface models (quadratic polynomial with all terms included) are constructed to relate the beam width and thickness to reliability index. Full factorial design of experiments with three levels is used to generate nine training points. The input variables (beam width and thickness) and the output

Table 10 The statistical properties of the random variables for the RBDO problem

Random variable	Distribution	Mean	Coefficient of variation
F_X (lb)	Normal	500	20 %
F_Y (lb)	Normal	1000	10 %
S (ksi)	Normal	40	5 %

Table 11 The response surface input and output variables for the RBDO problem

w (in)	t (in)	Analytical Rel. Ind.	Reliability index predictions of different methods			% difference compared to analytical value		
			Extrap. Bucher	Extrap. Proposed	SMCS	Extrap. Bucher	Extrap. Proposed	SMCS
2.2	3.6	0.386	0.497	0.387	0.385	28.8	0.3	−0.3
2.4	3.6	1.556	1.973	1.560	1.558	26.8	0.3	0.1
2.6	3.6	2.711	3.374	2.714	2.725	24.5	0.1	0.5
2.2	3.9	1.484	1.890	1.486	1.486	27.4	0.1	0.1
2.4	3.9	2.745	3.436	2.754	2.751	25.2	0.3	0.2
2.6	3.9	3.969	4.876	3.993	4.064	22.9	0.6	2.4
2.2	4.2	2.517	3.170	2.519	2.520	25.9	0.1	0.1
2.4	4.2	3.851	4.761	3.872	3.935	23.6	0.5	2.2
2.6	4.2	5.126	6.256	5.247	N/A[a]	22.0	2.4	N/A[a]

[a] mean value over 1000 repetitions cannot be computed because of the occurrence of reliability index prediction value of infinity due to limited sampling at high reliability levels

variable (reliability index) of the response surface models are provided in Table 11.

Table 11 also presents the comparison of the reliability index predictions of the proposed extrapolation formula to the Bucher's extrapolation formula and SMCS without extrapolation. Table 11 shows that Bucher's extrapolation formula leads to very large errors in reliability index predictions, even for small reliability levels. The extrapolation formula proposed in this paper outperforms SMCS without extrapolation for large reliability levels, whereas SMCS without extrapolation outperforms the proposed extrapolation formula for small reliability levels. Table 11 also shows that SMCS fails to provide reliability index prediction for the last training point that corresponds to a high reliability level (β=5.126). Conversely, the error of the proposed extrapolation formula is smaller than 3 % for all training points.

Response surface models are constructed for the analytical reliability index prediction, reliability index prediction using Bucher's extrapolation formula and the reliability prediction using the proposed extrapolation formula. The accuracies of the constructed response surface models are evaluated by using leave-one-out cross-validation errors. Response surface models are constructed nine times, each time leaving out one of the training points. The difference between the exact response at the omitted point and that predicted by each variant response surface model defines the cross-validation error. Table 12 provides the root mean square error (RMSE), the mean absolute error (MAE), the maximum absolute error (MAXE) as well as the mean and range of the response at training points. Comparison of these error metrics to the mean and range of response reveals that the constructed response surfaces are quite accurate.

After generating the response surface models, the RBDO problem is solved by using fmincon of MATLAB and following a multiple starting point strategy. The comparison of the metamodel based optimization results corresponding to the use of the analytical reliability index prediction, the reliability index prediction using Bucher's extrapolation formula and the reliability index prediction using proposed extrapolation formula to the actual optimum are provided in Table 13. It is found that the approximate optimum solution achieved through metamodel based optimization using analytical reliability index predictions is very close to the actual optimum, and the error in reliability index prediction is less than 0.1 %. This finding shows that the use of metamodel based optimization approach is an acceptable strategy for solving the RBDO problem of interest. It is also found that the approximate optimum solution achieved by the use of Bucher's extrapolation formula is not close to the actual optimum, and the corresponding error in reliability index is very large. Finally, it is also observed that

Table 12 Accuracies of the constructed response surface models evaluated using leave-one-out cross validation errors

Response	Mean of response	Range of response	RMSE[(a)]	MAE[(b)]	MAXE[(c)]
Analytical reliability index prediction	2.705	4.740	0.019	0.015	0.034
Reliability index prediction using Bucher's extrapolation formula	3.359	5.759	0.010	0.008	0.019
Reliability index prediction using the proposed extrapolation formula	2.726	4.860	0.022	0.018	0.044

[(a)] RMSE: root mean square error; [(b)] MAE: mean absolute error; [(c)] MAXE: maximum absolute error

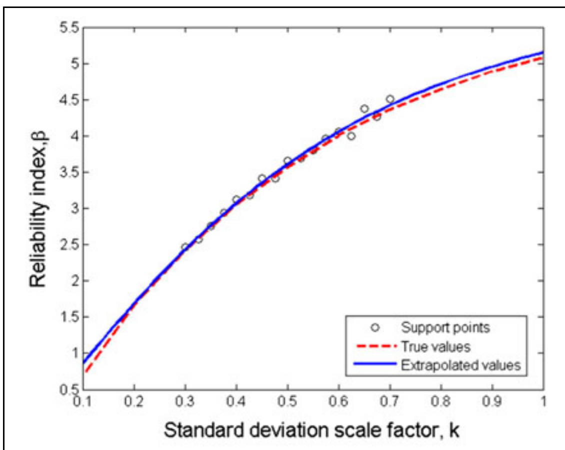
Table 13 Comparison of the metamodel based approximate optimums to the actual optimum

	w* (in)	t* (in)	Actual reliability index
Actual optimum	2.446	3.892	3.000
Metamodel based approximate optimums obtained using different reliability index estimation methods			
Analytical reliability index prediction	2.439	3.904	3.002
Reliability index prediction using Bucher's extrapolation formula	2.326	3.928	2.384
Reliability index prediction using the proposed extrapolation formula	2.435	3.907	2.994

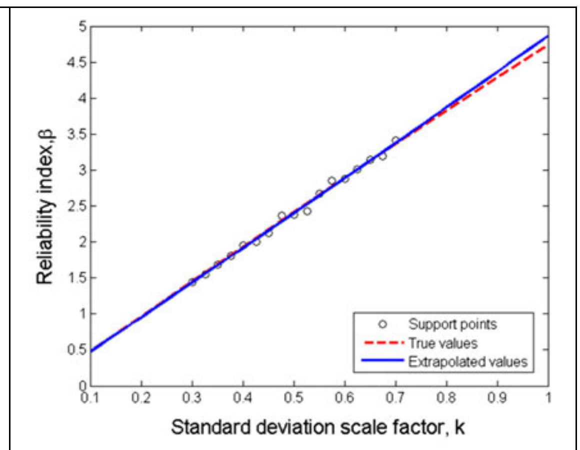
the approximate optimum solution achieved by the proposed method is very close to the actual optimum, and the corresponding error in reliability index is around 0.2 %.

4.6 Comparison of the true and extrapolated reliability index curves

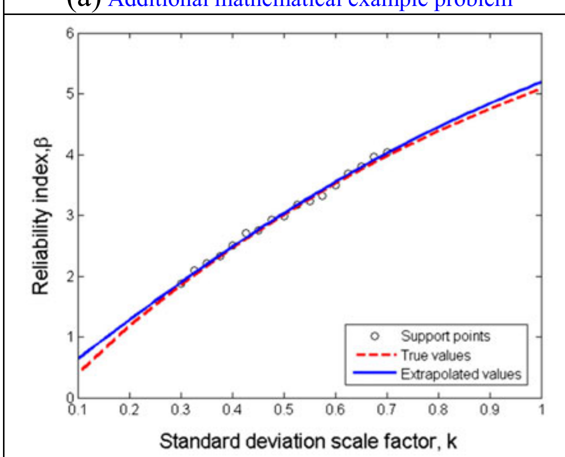
In Sections 4.1 through 4.5, the accuracy of the proposed extrapolation formula is evaluated by computing the error in the extrapolated reliability index compared to the true reliability index (evaluated through classical MCS with 10^9 samples). To provide a more comprehensive verification of the proposed extrapolation formula, this section presents comparison of the true and extrapolated reliability index curves in terms of the standard deviation scale factor, *k*. The true reliability index curves are obtained by computing the true reliability index for *k* values ranging from 0.1 to 1 by an increment of 0.1. The extrapolated reliability index curves are obtained following the procedure described in Section 3. The true and extrapolated



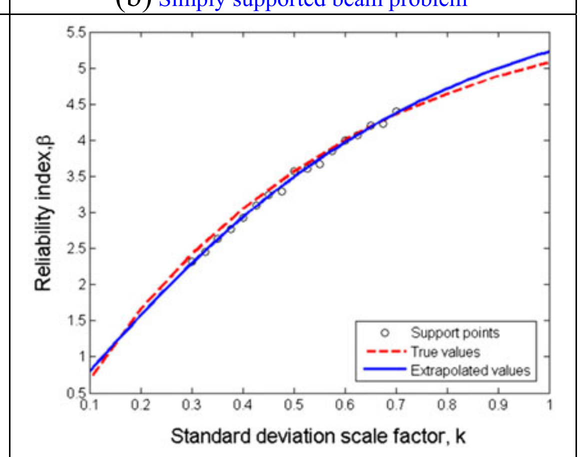
(a) Additional mathematical example problem



(b) Simply supported beam problem



(c) Torque arm problem



(d) RBDO problem

Fig. 9 The true and extrapolated reliability index curves reliability index curves are presented in Fig. 9 for all example problems except the illustrative example. For these example problems, the cases that lead to the largest reliability index value are considered. That is, the case of $\overline{X}_1 = 105$ is considered for the additional mathematical example, the case of $\overline{S} = 550,000$ is considered for the simply supported beam, the case of $\overline{S} = 210$ is considered for the torque arm, and the case of w= 2.6 in and t=4.2 in is considered for the RBDO problem.

5 Concluding remarks

The existing reliability index extrapolation approaches are based on using a standard deviation scale factor for all random variables, performing reliability index predictions for various values of scale factor, constructing a functional relationship between the scale factor and reliability index, and finally predicting the actual reliability through extrapolation via the functional relationship established. Since the standard deviation scale factor is applied to all random variables, the response function evaluation (often computationally expensive) needs to be performed for each value of the scale factor, therefore the computational cost is high. It is argued in this paper that if the limit state function has a separable form, the computational cost can be substantially reduced by eliminating the necessity of performing response function evaluation for each value of the scale factor.

A new reliability index extrapolation approach is proposed in this paper to predict the reliability of a highly safe structure that has a separable limit state function. By taking advantage of the separable form of the limit state function, the proposed extrapolation approach uses the standard deviation scale factor for the random variables that contributes only to the capacity function. Since the standard deviation of the random variables that affect the response function remains unchanged, the same set of realization of the response function can be used for any value of the scale factor used. Therefore, the proposed approach leads to a substantial reduction in the computational cost.

The accuracy of the proposed method is evaluated through mathematical and structural mechanics example problems. The performance of the extrapolation formula proposed in this paper is compared to an existing extrapolation formula and the separable Monte Carlo simulation method without extrapolation. It is found that the average performance of the proposed extrapolation formula is superior to the existing extrapolation formula, whereas the standard deviations of the reliability indices obtained from the existing formula is found to be smaller than the proposed extrapolation formula. It is also found that the performance of the proposed extrapolation method is better than the separable Monte Carlo simulation method without extrapolation at relatively high reliability indices, and the standard deviations of the reliability indices obtained from the proposed extrapolation formula is found to be smaller than separable Monte Carlo simulation method without extrapolation. In general, it is found that the proposed method can provide reasonable accuracy for high reliability index estimations with only 1000 response function evaluations. For the example problems investigated, the prediction of reliability indices up to β=5 could be performed with smaller than 4 % error with only 1000 response function evaluations.

Future research could focus on the following subjects:

- using bootstrap method to compute the conservative reliability estimates at the support points and exploring its effect on the extrapolated reliability index values
- comparison of the performance of the proposed approach to subset simulation
- developing a reliability index extrapolation formula for tail modeling

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