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RESEARCH PAPER

Approximate probabilistic optimization using exact-capacity-approximate-response-distribution (ECARD)

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Abstract Probabilistic structural design deals with uncertainties in response (e.g. stresses) and capacity (e.g. failure stresses). The calculation of the structural response is typically expensive (e.g., finite element simulations), while the capacity is usually available from tests. Furthermore, the random variables that influence response and capacity are often disjoint. In previous work we have shown that this disjoint property can be used to reduce the cost of obtaining the probability of failure via Monte Carlo simulations. In this paper we propose to use this property for an approximate probabilistic optimization based on exact capacity and approximate response distributions (ECARD). In Approximate Probabilistic Optimization Using ECARD, the change in response distribution is approximated as the structure is re-designed while the capacity distribution is kept exact, thus significantly reducing the number of expensive response simulations. ECARD may be viewed as an extension of SORA (Sequential Optimization and Reliability Assessment), which proceeds with deterministic optimization iterations. In contrast, ECARD has probabilistic optimization iterations, but in each iteration, the response distribution is approximated so as not to require additional response calculations. The use of inexpensive probabilistic optimization allows easy incorporation of system reliability constraints and optimal allocation of risk between failure modes. The method is demonstrated using a beam problem and a ten-bar truss problem. The former allocates risk between two different failure modes, while the latter allocates risk between members. It is shown that ECARD provides most of the improvement from risk re-allocation that can be obtained from full probabilistic optimization.

Keywords Reliability-based optimization · Probabilistic design · Uncertainty · Risk allocation · Approximation

Nomenclature

$\mathbf{x}_{r1}$	vector of those input random variables whose mean values cannot be controlled
$\mathbf{x}_{r2}$	vector of those input random variables whose mean values can be controlled
$\mathbf{u}$	vector of design variables
$\mathbf{u}_0$	vector of initial design variables
$\mathbf{u}_p$	vector of design variables at the end of the previous iteration
$g(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$	performance function
$c(\boldsymbol{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$	capacity (failure stress, allowable displacement, etc)
$r(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$	response (structural stress, maximum deflection, etc)
$r(\mu_{\mathbf{x}_{r1}}; \mu_{\mathbf{x}_{r2}}; \mathbf{u})$	response at the mean values of random variables
$r^{ECARD}(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$	approximate response used in ECARD iteration
$F_{Cj}(\bullet)$	cumulative distribution function of $c(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$ for jth mode

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$f_{Rj}(\bullet)$	probability density function of $r(\mathbf{x}_{r1};\mathbf{x}_{r2};\mathbf{u})$ for jth mode
P_{FSU}	upper bound of system probability of failure
P_{FS}	system probability of failure
$P_f^{\det}$	probability of failure at the deterministic optimum design
P_f^{ECARD}	approximate probability of failure
$P_{FSU}^{\det}$	upper bound of system probability of failure at the deterministic optimum design
$P_{FS}^{\det}$	system probability of failure at the deterministic optimum design
Δ	relative change in response
ε	convergence threshold

1 Introduction

Probabilistic structural design deals with uncertainties in responses (e.g. stresses) and capacities (e.g. failure stresses). The calculation of the structural response is typically expensive (e.g., finite element simulations), while the capacity is usually available from tests. Furthermore, the random variables that influence response and capacity are often disjoint; i.e., the random variables that affect response do not affect capacity and vice versa, and they are also statistically independent. Even if the performance function is not in disjoint form, i.e., we have random variables affecting both capacity and response; we can often separate the variables to achieve disjoint capacity and response.[1] This disjoint property was used to reduce the cost of obtaining the probability of failure via Separable Monte Carlo simulations proposed by Smarslok et al. (2006).

In this paper we propose to use this property for an approximate probabilistic optimization based on exact capacity and approximate response distributions (ECARD). In ECARD the change in response distribution is approximated as the structure is re-designed, while the exact distribution of capacity is used. It turns out that this approach can significantly reduce the number of expensive response simulations in calculating probability of failure.

ECARD may be viewed as an extension of sequential optimization and reliability assessment (SORA; Du and Chen 2004; Ba-abbad et al. 2006) methods. These methods reduce the cost of reliability calculations by stipulating the motion of the most probable point (MPP) as the structure is being re-designed so that the calculation of the reliability using FORM is trivial. ECARD may use similar assumptions for the response, but for the capacity, which is usually computationally cheaper, it does not make any assumptions. Like SORA, ECARD requires several iterations of optimization, with each iteration requiring a single accurate reliability assessment.

SORA methods do not allocate risk between failure modes in a structure where many components can fail simultaneously (Ba-abbad et al. 2006). Ba-abbad et al. (2006) proposed a modified SORA using FORM to reallocate risk between multiple failure modes, but this requires adding the individual failure modes as additional variables. It also does not cater to the correlation between various modes of failure. The objective of the present paper is to demonstrate that ECARD can provide near optimal risk allocation between failure modes at a fraction of the cost of full probabilistic optimization.

The remainder of the paper is organized as follows. Section 2 proposes an approximate method that allows probabilistic design based only on probability distribution of the capacity. The applications of the method to a beam problem and a ten-bar truss problem are presented in Sections 3 and 4, respectively. Finally, the concluding remarks are listed Section 5.

2 Approximate probabilistic optimization using exact-capacity-approximate-response-distribution (ECARD)

2.1 Approximation of change in response distribution

In probabilistic optimization, the system constraint is often given in terms of failure probability of a performance function. We consider here a specific form of performance function, given as

$$g\left(\mathbf{x};\mathbf{u}\right)=c\left(\mathbf{x}_c;\mathbf{u}\right)-r\left(\mathbf{x}_r;\mathbf{u}\right) \tag{1}$$

where $c(\mathbf{x}_c;\mathbf{u})$ and $r(\mathbf{x}_r;\mathbf{u})$ are capacity and response, respectively. Both the capacity and response are random because they are functions of random variables $\mathbf{x}_c$ and $\mathbf{x}_r$, respectively, and both may depend on design variables $\mathbf{u}$. The system is considered to be failed when the response exceeds the capacity. We assume that the probability distribution of $c(\mathbf{x}_c;\mathbf{u})$ is well known, while that of $r(\mathbf{x}_r;\mathbf{u})$ requires a large number of analyses to define. The separable form in (1) is used here

[1] For instance, if performance function is $g\left(r,c\right)=r/(1+c)-c$, then we can multiply the equation by $(1+c)$ and make the function disjoint with new capacity of $c(1+c)$. However, we recognize that there are exceptions like when we have size effect on strength or implicit, "black-box" transfer functions.

for simplicity. However, as explained later, the only requirements are that $\mathbf{x}_c$ and $\mathbf{x}_r$ are independent, and that the calculation of the limit state $g(\mathbf{x}, \mathbf{u})$ is inexpensive given r and c. Furthermore, we can separate random variables, $\mathbf{x}$, into those variables whose mean values cannot be controlled, $\mathbf{x}_{r1}$, and those variables whose mean values can be controlled, $\mathbf{x}_{r2}$. Therefore, vector $\mathbf{u}$ contains *only the remaining deterministic design variables*. This distinction facilitates the approximation of response distribution by measuring changes in response due to changes in mean values of random variables that can be controlled, $\mathbf{x}_{r2}$.

Since the capacity and response are independent, the probability of failure jth mode may be calculated as (Melchers 2002a)

$$P_{fj} = \Pr\left[g_j\left(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u}\right) \leq 0\right] = \int_{-\infty}^{\infty} F_{Cj}\left(\xi\right) f_{Rj}\left(\xi\right) d\xi. \tag{2}$$

In the above equation, $F_{Cj}(\xi)$ is the cumulative distribution function (CDF) of capacity for jth mode, and $f_R(\xi)$ is the probability density function (PDF) of response for the same mode. When $F_{Cj}(\xi)$ is given, evaluation of the integral using Monte Carlo simulation (MCS) of the response is known as the conditional MCS. That is, with N simulations of the response system probability of failure is given by (3), where 'm' is number of modes of failure. This equation is based on the fact that for a given response the capacities for the modes are independent.

$$P_f = \frac{1}{N}\sum_{i=1}^{N}\left[1 - \prod_{j=1}^{m}\left\{1 - F_{Cj}\left(r_i\right)\right\}\right] \tag{3}$$

Smarslok et al. (2006) showed that the conditional MCS is much more accurate than the traditional MCS where (1) is used. For a more general limit state, it is still possible to use a similar approach, called separable MCS (Smarslok et al. 2006).

When design variables are changed, the distributions of both capacity and response may change. We consider the case that it is inexpensive to update the distribution of the capacity but it is expensive to update the distribution of the response. There are different ways to update the distribution of the response. For instance, Nikolaidis et al. (2008) proposed a method to update the system failure probability when the mean values of random variables change according to redesign and there are no other changes. However, here we use the simplest possible approach, with the aim of showing that even with a very crude approximation it can provide reasonable risk allocation between failure modes.

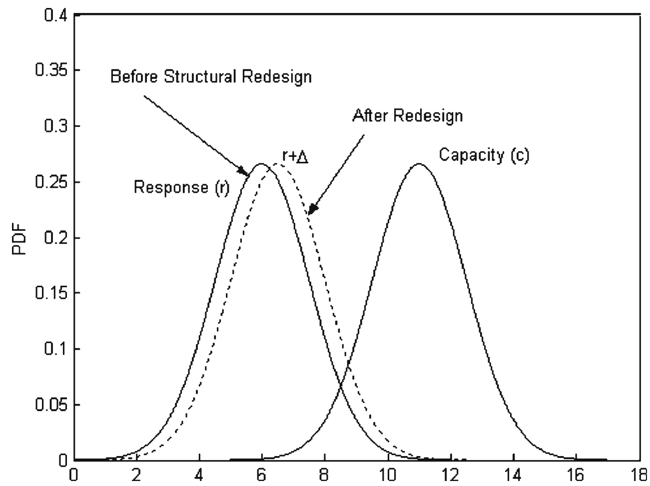

Fig. 1 Distributions of response before and after redesign. Redesign only changes the mean value. Distribution of capacity is also shown

In ECARD, the change in response distribution is represented by the change in the mean value μ_{xr2} of the random variable $\mathbf{x}_{r2}$ and changes in $\mathbf{u}$. That is, if $r(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u}_0)$ is the response distribution at a given design $\mathbf{u}_0$, then the response distribution at a new design $\mathbf{u}$, can be approximated by

$$r\left(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u}\right) = r\left(\mathbf{x}_{r1}; \mu_{\mathbf{x}_{r2_0}}; \mathbf{u}_0\right) + \Delta;$$
$$\Delta = r\left(\mu_{\mathbf{x}_{r1}}; \mu_{\mathbf{x}_{r2}}; \mathbf{u}\right) - r\left(\mu_{\mathbf{x}_{r1}}; \mu_{\mathbf{x}_{r2_0}}; \mathbf{u}_0\right) \tag{4}$$

Figure 1 illustrates the change in response distribution, along with the distribution of capacity. The change in response distribution is approximated such that the standard deviation of the response remains constant in (4). This assumption may be acceptable when the changes in design variables are small. We start the procedure with the deterministic design. Since the probabilistic optimum design can usually be found close to the deterministic optimum design, the above assumption will not cause large error. In addition, since ECARD recalculates $r(\mathbf{x}_{r1}; \mathbf{x}_{r2}; \mathbf{u})$ at every ECARD iteration, the error will be reduced toward the probabilistic optimum design. This simple approximation may not always work and should be replaced with more sophisticated approximations if needed. In fact convergence of ECARD iterations is not guaranteed.[2] But even this simple minded approximation worked

[2]We know that SORA developers faced convergence problems and this algorithm may also jump between feasible and infeasible regions for some problems.

very well with ECARD for the two examples shown here.

Another important aspect of the proposed ECARD is that it directly perturbs the distribution of output response, rather than input design variables. This contributes the significant efficiency of the proposed method because it does not require calculating propagation of input uncertainty to the output response distribution. The new response distribution in (4) can be obtained with a single deterministic analysis of $r(\mathbf{x}_{r1};\mathbf{x}_{r2};\mathbf{u})$.

We have also considered a similar approximation where the response is scaled by $1+\Delta/r\left(\mathbf{x}_{r1};\mathbf{x}_{r2};\mathbf{u}_0\right)$ so that the coefficient of variation is preserved in Acar et al. (2007). The difference in performance was not significant.

The change in the response distribution will change the probability of failure. Combining (3) and (4), the probability of failure at the new design is approximated by

$$P_f^{\text{ECARD}} = \frac{1}{N}\sum_{i=1}^{N} F_C\left(r_i+\Delta\right) \tag{5}$$

2.2 Approximate probabilistic optimization using ECARD

In this section the ECARD optimization process is explained. The goal is to further reduce the objective function from the deterministic optimal design, while maintaining the same level of probability of failure. This can be achieved by re-allocating risk between either different failure modes or different structural members. The process takes the following steps:

1. Perform deterministic optimization with safety factors to obtain a deterministic optimum design $\mathbf{u}_{\text{det}}$ and objective function W_{det}.
2. Set the initial ECARD design $\mathbf{u}_0=\mathbf{u}_{\text{det}}$. The probabilistic design starts from the deterministic optimum design; $\mathbf{u}_0$. Set $p=0$. Perform conditional MCS to generate distribution of response $r(\mathbf{x}_r;\mathbf{u}_0)$ and its probability of failure, $P_f^{\text{det}}=P_f^0$. It is possible to use the First-Order Reliability Method (FORM) instead of MCS (see Acar et al. 2007).
3. At the current design, $\mathbf{u}_p$, calculate the response at the mean value of the random variables, $r_\mu = r\left(\mu_X;\mathbf{u}_p\right)$.
4. Obtain optimum design $\mathbf{u}_{\text{opt}}$ and optimum objective function W_{opt} by solving the following deterministic optimization problem with target probability $P_f^{\text{det}}=P_f^0$:

$$\begin{aligned}&\underset{\mathbf{u}}{\text{Minimize}} && W\left(\mu_{\mathbf{X}},\mathbf{u}\right)\\ &\text{s.t.} && P_f^{\text{ECARD}} \le P_f^{\text{det}}\end{aligned} \tag{6}$$

where

$$\Delta = r\left(\mu_{\mathbf{x}_{r1}};\mu_{\mathbf{x}_{r2}};\mathbf{u}\right) - r\left(\mu_{\mathbf{x}_{r1}};\mu_{\mathbf{x}_{r2_0}};\mathbf{u}_0\right)$$

$$r^{ECARD}\left(\mathbf{x}_{r1};\mathbf{x}_{r2};\mathbf{u}\right) = r\left(\mathbf{x}_{r1};\mu_{\mathbf{x}_{r2_0}};\mathbf{u}_0\right)+\Delta$$

$$P_f^{\text{ECARD}} = \frac{1}{N}\sum_{i=1}^{N} F_C\left(r_i^{\text{ECARD}}\right) \tag{7}$$

5. Perform MCS to generate accurate distribution of response $r(\mathbf{x}_r;\mathbf{u}_{\text{opt}})$ and its probability of failure P_f, at $\mathbf{u}_{\text{opt}}$. This step requires full reliability analysis, which is the expensive part.
6. Check accuracy and convergence: The ECARD approximation is considered accurate when $\left|P_f^{\text{ECARD}}-P_f\right|\le\varepsilon$, and convergence is assumed when the changes in design variables are small. If the process is accurate and converged, stop the process. Otherwise, set $p=p+1$, $\mathbf{u}_p=\mathbf{u}_{\text{opt}}$ and go to Step 3.

The accuracy of ECARD to locate the true optimum depends on the magnitudes of errors involved in the approximations. The simple approximation used here is accurate if changes in distribution parameters of the response due to redesign are small. In addition, the accuracy in estimating the probability of failure affects the convergence rate of the proposed method. However, more accurate approximations such as Nikolaidis et al. (2008) can be used to improve accuracy.

Note that (7) is easily applicable to a single mode of failure. For multiple modes of failure we need to use the joint distribution of the capacities to calculate the system failure probabilities for a vector of response quantities. In the examples used in this paper we use instead the Ditlevsen upper bound (Melchers 2002b). However, the proposed method should work even

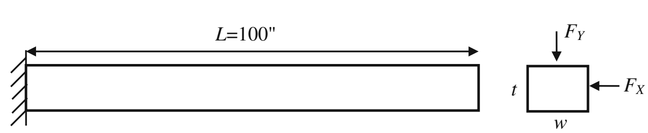

Fig. 2 Cantilever beam: geometry and loadings

Table 1 The mean and coefficient of variation of the random variables **x**

Random variable	Mean	Coefficient of variation
F_X (lb)	500	20%
F_Y (lb)	1,000	10%
E (psi)	2.9E7	5%
σ_f (psi)	40,000	5%

All variables follow normal distribution

with the more accurate system probabilities obtained from MCS.

3 Application of ECARD to beam optimization problem

Deterministic optimization based on safety factors typically may have sub-optimal risk allocation between failure modes compared to probabilistic optimization. In this section, design optimization of a cantilever beam is presented in order to show the risk allocation capability of ECARD optimization. In order to gage the effectiveness of ECARD compared to full probabilistic optimization, we compare both to the deterministic optimization.

3.1 Problem description

The cantilever beam design problem (Fig. 2) has been analyzed by many researchers including Wu et al. (2001), Qu and Haftka (2004), and Ba-abbad et al. (2006). The beam has two failure modes: stress failure and excessive displacement. The minimum weight design is sought by varying the width w and thickness t of the beam; i.e., $\mathbf{u} = [w, t]$. The applied loads F_X and F_Y along with the elastic modulus E and failure stress σ_f are random variables. All random variables are assumed normally distributed with mean and coefficient of variation as listed in Table 1. The beam width w and thickness t are modeled as deterministic variables.

The performance function corresponding to stress failure mode can be written as

$$g_1 = \sigma_f - \left(\frac{600}{wt^2} F_Y + \frac{600}{w^2 t} F_X \right) \equiv c_1 (\mathbf{x}) - r_1 (\mathbf{x}; \mathbf{u}), \quad (8)$$

where c_1 and r_1 are the capacity and response of g_1. Similarly, the performance function corresponding to displacement failure mode can be written as

$$g_2 = \frac{D_0 E}{4L^3} - \frac{1}{wt} \sqrt{\left(\frac{F_Y}{t^2} \right)^2 + \left(\frac{F_X}{w^2} \right)^2} \equiv c_2 (\mathbf{x}) - r_2 (\mathbf{x}; \mathbf{u}), \quad (9)$$

where c_2 and r_2 are the capacity and response of g_2, and D_0 (here 2.2535″) is the critical displacement. We rearranged the performance functions so that the capacity is independent of design variables. The probabilities of failure, P_{f1} and P_{f2}, can be calculated using the conditional MCS in (5).

3.2 Deterministic optimization

If the deterministic optimization is performed with the two performance functions in (8) and (9), the probability of failure at the optimum design will be close to 50%. The deterministic design takes into account the effect of random variables using a safety factor. We employ a safety factor, $S_F = 1.5$, to the applied loads that is typical of aerospace structural design. Then the deterministic optimization problem for minimum weight can be written as

$$\begin{aligned} &\underset{w,t}{\text{Minimize}} && A = w\,t \\ &\text{s.t.} && c_1 - S_F r_1 \geq 0 \\ &&& c_2 - S_F r_2 \geq 0 \end{aligned} \quad (10)$$

The deterministic optimization problem in (10) is solved using the Sequential Quadratic Programming algorithm in MATLAB (using the function *fmincon*). The results of deterministic optimization are listed in Table 2. The probabilities of failure are calculated using conditional MCS. The probabilities of failure for stress and displacement constraints are denoted by P_{f1}^{det} and P_{f2}^{det}, respectively. We use one million samples for the conditional MCS. With the traditional MCS, the coefficient of variation associated with probability of failure of stress constraint would have been 10% and 2% for displacement constraint. Numerical experiments showed that with the conditional MCS the corresponding coefficients of variation are 3% and 1%,

Table 2 Deterministic optimum results of the cantilever beam problem

	w (in)	t (in)	A (in^2)	P_{f1}^{det}	P_{f2}^{det}	P_{FSU}^{det}
Conditional MCS	2.275	4.414	10.042	9.82E–05	2.66E–03	2.76E–03

The subscript 1 corresponds to stress failure and the subscript 2 to displacement failure

Table 3 Comparison of system probability of failure P_{FS}^{det} and its upper bound P_{FSU}^{det} for the deterministic optimum of cantilever beam problem using one million traditional MCS

	P_{f1}	P_{f2}	P_{FSU}^{det}	P_{FS}^{det}	$\lvert P_{FSU}^{\mathrm{det}} - P_{FS}^{\mathrm{det}}\rvert$
Traditional MCS	8.50E–05	2.61E–03	2.69E–03	2.66E–03	3.90E–05

Small differences with Table 2 are due to the use of traditional MCS. The subscript 1 corresponds to stress failure and the subscript 2 to displacement failure

respectively. We used MCS with one million samples to calculate also the system probability of failure P_{FS}^{det} and compare it in Table 3 to the Ditlevsen's first-order upper bound P_{FSU}^{det} that estimates the system failure probability as the sum of the individual failure probabilities (Melchers 2002b). The traditional MCS has poorer accuracy than the conditional MCS, so the numbers are slightly different. However, they still show that the failure modes are well correlated because the Ditlevsen approximation error is close to half of the lower probability of failure (see Table 3).

Note that the deterministic optimization allocates risk between two failure modes such that the probability of failure for the displacement mode is 27 times higher (see values from Table 2) than that of the stress mode. This is based on the use of the same safety factor for both failure modes. Its contributions to the different failure modes are different. We will show that the probabilistic optimization reduces the difference in failure probabilities significantly in the following section.

3.3 Probabilistic optimization

The probabilistic optimization is formulated with the same objective function. However, the constraint is imposed on the approximate (upper bound) system probability of failure. The goal is to reduce the objective function further by reallocating the risk between the two failure modes, while the system probability of failure should not be greater than that of the deterministic optimization. The probabilistic optimization problem is then formulated as

$$\begin{aligned} &\underset{w,t}{\text{Minimize}} \quad A = w\,t \\ &\text{s.t.} \qquad P_{FS} = \left(P_{f1} + P_{f2}\right) \leq P_{FSU}^{\mathrm{det}} \end{aligned} \tag{11}$$

The results of probabilistic optimization are listed in Table 4. The probabilistic optimization is computationally expensive, requiring 61 reliability assessments. As can be seen from the table, the design obtained at the end of the optimization satisfied the probability constraint with a small margin. Compared with the deterministic optimum (Table 2), the probabilistic optimization reduced the weight by 6%, while reducing the system failure probability by 2%. The reduction in both the weight and the system probability of failure is obtained by risk reallocation between two failure modes. The deterministic design leads to smaller probability of failure for the stress mode than that of the displacement mode, while the situation is reversed for the probabilistic design. This is based on the fact that displacement is cheaper to control than stress in terms of weight expenditure. For example, if w and t are scaled uniformly, then the stress is proportional to the cube of the scale factor while the displacement is proportional to the 4th power. Similar results are also reported by Ba-abbad et al. (2006) using a modified SORA. Table 5 compares the system probability of failure and its upper bound used in probabilistic optimization procedure. It is seen that again the error is a substantial fraction of the lower of the two probabilities of failure, indicating substantial correlation.

3.4 Approximate probabilistic optimization using ECARD

The approximate probabilistic optimization problem is formulated based on (11) by replacing the probability of failure with approximate one, as

$$\begin{aligned} &\underset{w,t}{\text{Minimize}} \quad A = wt \\ &\text{s.t.} \qquad P_{FSU}^{\mathrm{ECARD}} = \left(P_{f1}^{\mathrm{ECARD}} + P_{f2}^{\mathrm{ECARD}}\right) \leq P_{FSU}^{\mathrm{det}} \end{aligned} \tag{12}$$

Table 4 Probabilistic optimum results of the cantilever beam problem

	w (in)	t (in)	A (in^2)	P_{f1}	P_{f2}	P_{FS}
Conditional MCS	2.651	3.559	9.437	2.37E–3	3.31E–4	2.70E–3

The subscript 1 corresponds to stress failure and the subscript 2 to displacement failure

Table 5 Comparison of system probability of failure P_{FS} and its bound P_{FSU} for the probabilistic optimum of cantilever beam problem using one million traditional MCS

	P_{f1}	P_{f2}	P_{FSU}	P_{FS}	$\lvert P_{FSU} - P_{FS}\rvert$
Crude MCS	2.38E–03	3.35E–04	2.72E–03	2.61E–03	1.10E–04

Small differences with Table 4 are due to the use of the traditional MCS. The subscript 1 corresponds to stress failure and the subscript 2 to displacement failure

where P_{f1}^{ECARD} and P_{f2}^{ECARD} are, respectively, approximations of P_{f1} and P_{f2} using the ECARD method described in Section 2.2. The approximate probabilities of failure are calculated as

$$P_{f1}^{\mathrm{ECARD}} = \frac{1}{N}\sum_{i=1}^{N} F_{C1}\left(r_1^i + \Delta_1\right)$$

$$P_{f2}^{\mathrm{ECARD}} = \frac{1}{N}\sum_{i=1}^{N} F_{C2}\left(r_2^i + \Delta_2\right) \tag{13}$$

where F_{C1} and F_{C2} are CDFs of the two capacities. The optimization in (12) is solved iteratively until the convergence criterion in Section 2.2 is satisfied. We have used a gradient based method, SQP implemented by MATLAB's *fmincon* function. However, in general any optimization method can be used.

Figure 3 shows the deterministic and probabilistic optimum designs, along with the iteration history of the ECARD design. The contour lines of $P_{FSU} = 0.0027$ and $A = 9.4356$ in^2 are shown for reference. It is seen that the approximate ECARD probability of failure at iteration 1 has an error compared to the accurate one. However, it is gradually reduced in the following iterations. Table 6 lists the designs obtained during iterations of the ECARD optimization using conditional MCS. After the 4th iteration, the ECARD optimization was considered converged because the design variables changed by less than 0.1% and the approximate and exact probabilities of failure were close. The ECARD design was close to the probabilistic optimum. The difference between approximate and full probabilistic optimizations can be explained by the convergence tolerance ε, error in probability estimation, and small change in objective function near the optimum design. However, in general, in the limit the error may not be negligible or it may take more number of iterations to converge depending on the tightness of convergence criteria. In fact we might need better approximation to deal with the issue of accuracy and efficiency.

Table 7 compares the system probability and its upper bounds for the ECARD design. Comparing to the probabilistic optimum, the ECARD optimum is about 0.3% heavier, while having the same system probability of failure. As in the probabilistic optimization, this is achieved by increasing the risk of the more expensive (in terms of weight) stress mode and decreasing the risk of the cheaper displacement mode. Note that ECARD requires only four reliability assessments as opposed to 61 used by full probabilistic optimization. Every ECARD iteration needs a reliability assessment in order to correct errors introduced by the approximations involved in the response distribution. In the first two iterations, steps are larger and accuracy is not good. It improves in subsequent iterations.

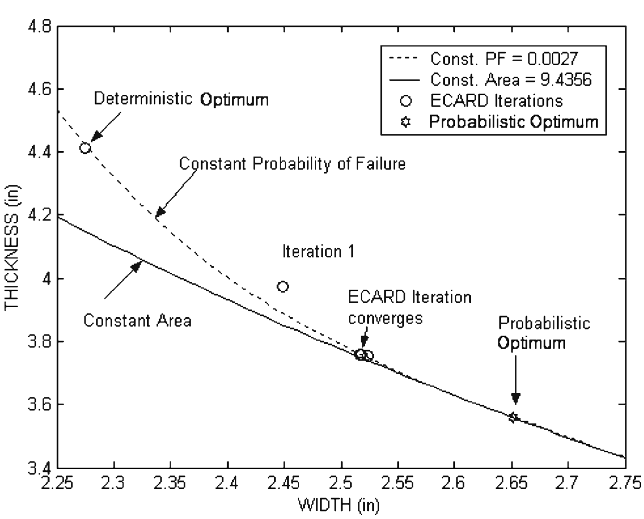

Fig. 3 ECARD design history with deterministic and probabilistic designs

4 Application of ECARD to ten-bar truss problem

The second example is a ten-bar truss problem, as shown in Fig. 4. First, we present deterministic optimization of the problem. Then, probability of failure calculation using MCS is discussed. Finally, probabilistic optimization is performed using ECARD, and the accuracy and efficiency of the method is evaluated.

Table 6 Iterations of approximate ECARD probabilistic optimization for the cantilever beam problem using conditional MCS

Iter.	w (in)	t (in)	A (in^2)	P_{f1}	P_{f1}^{ECARD}	P_{f2}	P_{f2}^{ECARD}	P_{FSU}	P_{FSU}^{ECARD}
0	2.2752	4.4137	10.042	9.56E–05	9.56E–05	2.76E–03	2.76E–03	2.86E–3	2.70E–3
1	2.4490	3.9723	9.7281	4.00E–04	4.00E–04	6.00E–04	2.30E–03	1.00E–3	2.70E–3
2	2.5234	3.7561	9.4783	1.70E–03	1.50E–03	7.00E–04	1.20E–03	2.40E–3	2.70E–3
3	2.5162	3.7569	9.4530	1.90E–03	1.90E–03	9.00E–04	8.00E–04	2.80E–3	2.70E–3
4	2.5180	3.7582	9.4630	1.80E–03	1.80E–03	8.00E–04	9.00E–04	2.60E–3	2.70E–3

4.1 Problem description

The problem definition for the ten-bar truss problem is taken from Haftka and Gurdal (1992) (page 237). The ten-bar truss structure in Fig. 4 is under two loads, P_1 and P_2. The design goal is to minimize the weight, W, by varying the cross-sectional areas, A_i, of the truss members. The design should satisfy stress constraints and minimum gage constraints. Input data are summarized in Table 8. In the traditional deterministic design of aircraft structures, uncertainty in material failure stress is taken into account using a knockdown factor, with so-called A-basis or B-basis properties (Little 2003). The relationship between allowable stress and the mean value of failure stress is

$$\sigma_{\text{allowable}} = K_{dc}\sigma_f \tag{14}$$

where K_{dc} is the knockdown factor. In this example, A-basis allowable stress is used in which the knockdown factor becomes $K_{dc} = 0.87$. In addition, uncertainties in applied loads and errors in stress calculations are considered by multiplying the applied loads by a safety factor of 1.5.

4.2 Deterministic optimization

The deterministic optimization protects against uncertainties using safety factor and knockdown factor. The deterministic optimization problem can be formulated as

$$\begin{aligned} \min_{A_i} \quad & W = \sum_{i=1}^{10} \rho L_i A_i \\ \text{s.t.} \quad & \frac{N_i\left(S_F P_1, S_F P_2, \mathbf{A}\right)}{A_i} \\ & = \sigma_i \le (\sigma_{\text{allowable}})_i\,, \quad i = 1, \ldots, 10 \end{aligned} \tag{15}$$

where L_i, N_i, and A_i are, respectively, the length, member force, and cross-sectional area of element i. $\mathbf{A}$ is the vector of cross-sectional areas, σ_i and $(\sigma_{\text{allowable}})_i$ are the stress and allowable stress of an element, respectively. In this example, $(\sigma_{\text{allowable}})_i$ corresponds to the capacity, while σ_i to the response. The applied loads are multiplied by a safety factor in order to consider the effects of uncertain parameters. The analytical solutions for the member forces are given in Appendix. The results of deterministic optimization are listed in Table 9. It is noted that Element 5 is a zero-force member and the cross-sectional areas of Elements 2, 5, and 6 reach the minimum gage.

The probabilities of failure of the elements, given in the second last column of Table 9, are calculated using the conditional MCS discussed in the next section. Last column of Table 9 presents component probability of failure calculations using 10 million samples of traditional MCS, and it is similar to the values obtained with conditional MCS. However, the system failure (2.90E–02) in the last row is significantly lower than the upper bound obtained by adding elemental probabilities of failure (4.03E–02). It is noted that three elements (2, 6, and 10) contribute about 80% of the system probability of failure (approximated with the Ditlevsen bounds). Since Elements 2 and 6 are at minimum gage and Element 10 is close to it, it is easy to reduce the system probability of failure by reallocating small additional weights to these members. However, the deterministic optimization did not do it because it applies uniform safety factors and knockdown factors to all members.

4.3 Calculation of probability of failure using conditional MCS

Many uncertainties are involved in the design of ten-bar truss, such as variability from manufacturing, loading,

Table 7 Probability of failure of system at last iteration of ECARD for cantilever beam problem using one million traditional MCS

	P_{f1}	P_{f2}	P_{FSU}	P_{FS}	$\lvert P_{FSU} - P_{FS}\rvert$
Traditional MCS	1.85E–03	7.97E–04	2.65E–03	2.47E–03	1.80E–04

The subscript 1 corresponds to stress failure and the subscript 2 to displacement failure

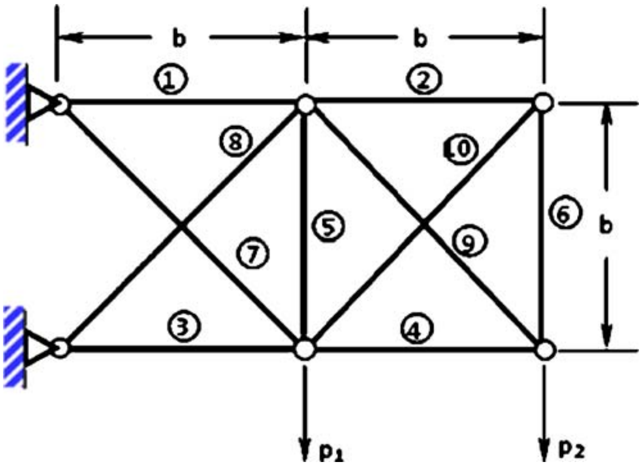

Fig. 4 Geometry and loadings of the ten-bar truss example

and material properties, and errors from modeling and numerical calculation. In the following, we will detail these uncertainties.

Failure of an element is predicted to occur when the stress in the element is larger than its failure stress. That is, the performance function for an element can be written as

$$g = \left(\sigma_f\right)_{\text{true}} - \sigma_{\text{true}} \tag{16}$$

where the subscript 'true' stands for the true value of the relevant quantity, which is different from its calculated (or predicted) value due to errors. Introducing errors, (16) can be re-written as

$$g = \left(1 - e_f\right)\sigma_f - \left(1 + e_\sigma\right)\sigma \tag{17}$$

Here, e_f is the error in failure stress prediction (e.g., due to size effects), e_σ is the error in stress calculation (e.g., due to errors in mathematical model). We formulated the errors such that positive errors correspond to a conservative decision. Hence, the sign in front of error in stress is positive, while that in failure stress is negative. Even though our stress calculation is exact, we pretend that we have error, e_σ, considering the analysis of a more complex structure where the stresses are calculated using Finite Element Analysis (FEA). The calculated stress can be written in a compact form as

$$\sigma = \sigma_{\text{FEA}}\left[\left(1 + e_{P1}\right)P_1, \left(1 + e_{P2}\right)P_2, \left(1 + \mathbf{e_A}\right)\mathbf{A}\right] \tag{18}$$

where $\sigma_{\text{FEA}}[\bullet]$ stands for calculated stresses using FEA, e_{P1} and e_{P2} are errors in loads P_1 and P_2, and $\mathbf{e_A}$ is the vector of errors corresponding to 10 cross-sectional areas. The performance function for Element i can be arranged in a form (i.e., in a form that allows the use of conditional MCS) as

$$\begin{aligned} g_i &= \left(\sigma_f\right)_i - \frac{\left(1 + e_\sigma\right)}{\left(1 - e_f\right)}\sigma_{\text{FEA}} \\ &\quad \times \left[\left(1 + e_{P1}\right)P_1, \left(1 + e_{P2}\right)P_2, \left(1 + \mathbf{e}_A\right)\mathbf{A}\right]_i \\ &\equiv c_i - r_i \end{aligned} \tag{19}$$

where c_i and r_i are, respectively, the capacity and response.

In addition to the errors, variability is also present in the performance function through σ_f, P_1, P_2, and $\mathbf{A}$, which are modeled as random variables. These errors and variability are considered random variables. The distribution types and probabilistic parameters of errors and variability are listed in Table 10.

The probabilities of failure of the elements are calculated using conditional MCS in (5). The system probability of failure is again approximated by Ditlevsen's first-order upper bound

$$P_{FSU} = \sum_{i=1}^{10} \left(P_f\right)_i \tag{20}$$

4.4 Probabilistic optimization

Starting from the deterministic optimum design, the probabilistic optimization problem can be formulated such that the weight of the structure is minimized (calculated using mean values of the areas), while maintaining the same system probability of failure with that of the deterministic optimum design. Thus, we have

$$\begin{aligned} &\underset{\overline{A}_i}{\text{Minimize}} \quad W = \sum_{i=1}^{10} \rho L_i \overline{A}_i \\ &\text{s.t.} \qquad P_{FSU} \leq P_{FSU}^{\text{det}} \end{aligned} \tag{21}$$

where P_{FSU} is the upper bound on the system probability of failure. The design variables are the mean values

Table 8 Input data for ten-bar truss problem

Parameters	Values
Dimension, b	360 in
Safety factor, S_F	1.5
Load, P_1	66.67 kips
Load, P_2	66.67 kips
Knockdown factor, K_{dc}	0.87
Density, ρ	0.1 lb/in^3
Modulus of elasticity, E	10^4 ksi
Allowable stress, $\sigma_{\text{allowable}}$	25 ksi[a]
Minimum gage	0.1 in^2

[a]For Member 9, allowable stress is 75 ksi

Table 9 Results of deterministic optimization of the ten-bar truss problem

Element	$A_i^{\text{det}}(\text{in}^2)$	W_i (lb)	Stress (ksi)	Conditional MCS $\left(P_f^{\text{det}}\right)_i$	10^7 Samples of traditional crude MCS $\left(P_f^{\text{det}}\right)_i$
1	7.900	284.4	25.0	2.13E–03	2.10E–03
2	0.100	3.6	25.0	1.06E–02	1.05E–02
3	8.100	291.6	−25.0	4.80E–04	5.00E–04
4	3.900	140.4	−25.0	2.19E–03	2.30E–03
5	0.100	3.6	0.0	4.04E–04	4.00E–04
6	0.100	3.6	25.0	1.07E–02	1.04E–02
7	5.798	295.2	25.0	1.69E–03	1.70E–03
8	5.515	280.8	−25.0	1.89E–03	1.80E–03
9	3.677	187.2	37.5	5.47E–13	0.00E+00
10	0.141	7.2	−25.0	1.07E–02	1.06E–02
Total	–	1,497.6	–	4.08E–02	4.03E–02
System probability of failure					2.90E–02

The probability of failure is calculated using conditional MCS with 10^6 samples

Table 10 Probabilistic distribution types and parameters for errors and variability in the ten-bar truss problem

Uncertainties	Distribution type	Mean	Scatter
Errors			
e_σ	Uniform	0.0	±5%
e_{P1}	Uniform	0.0	±10%
e_{P2}	Uniform	0.0	±10%
$\mathbf{e}_{\mathrm{A}}$ (10 × 1 vector)	Uniform	0.0	±3%
e_f	Uniform	0.0	±20%
Variability			
P_1, P_2	Extreme type I	66.67 kips	10% c.o.v.
A (10 × 1 vector)	Uniform	$\overline{\mathrm{A}}$ (10 × 1 vector)	±4%
σ_f	Lognormal	$25/K_{dc}$ ksi or $75/K_{dc}$ ksi	8% c.o.v.

Table 11 Results of probabilistic optimization of the ten-bar truss problem using conditional MCS with 10,000 samples

Elements	A_i^{det}	A_i	Mean Stresses in ksi at A_i	Conditional MCS $\left(P_f^{\text{det}}\right)_i$	Conditional MCS $\left(P_f\right)_i$	10^7 Samples crude MCS $\left(P_f\right)_i$
1	7.900	7.1920	18.349	2.14E–03	5.88E–03	5.80E–03
2	0.100	0.3243	15.675	1.04E–02	3.07E–03	3.00E–03
3	8.100	7.1620	−19.329	5.07E–04	8.26E–03	8.30E–03
4	3.900	3.7010	−16.965	2.41E–03	2.15E–03	2.10E–03
5	0.100	0.4512	04.434	3.66E–04	3.18E–05	0.00E+00
6	0.100	0.3337	15.287	1.07E–02	2.14E–03	2.00E–03
7	5.798	5.1697	19.492	1.56E–03	1.02E–02	1.05E–02
8	5.515	4.9782	−18.286	1.92E–03	3.75E–03	3.80E–03
9	3.677	3.5069	25.302	4.10E–13	4.70E–13	0.00E+00
10	0.141	0.4325	−16.579	1.06E–02	5.46E–03	5.90E–03
Total	1,497.6 lb	1,407.13 lb		4.10E–02	4.10E–02	4.14E–02
System probability of failure using crude MCS						3.37E–02

Table 12 Results of approximate probabilistic optimization and progress toward the accurate optimum

Element	Determ. des.	iter_01	iter_02	iter_03	iter_04	iter_05
Areas (in^2)						
1	7.900	7.2383	7.4793	7.3684	7.4215	7.395
2	0.100	0.100	0.100	0.100	0.100	0.100
3	8.100	6.8277	7.2051	7.0629	7.1187	7.0944
4	3.900	3.9645	3.9954	3.9511	3.9842	3.9636
5	0.100	0.100	0.100	0.100	0.100	0.100
6	0.100	0.100	0.100	0.100	0.100	0.100
7	5.798	4.9161	5.2133	5.0925	5.1422	5.1198
8	5.515	5.2035	5.3075	5.2562	5.2834	5.2688
9	3.677	3.9924	4.0504	3.9732	4.0212	3.9953
10	0.141	0.1256	0.1364	0.1296	0.1336	0.1311
Weight (lb)	1,497	1,384	1,432	1,408	1,420	1,414
Mean stresses (ksi)						
1	16.667	18.294	17.672	17.949	17.817	17.882
2	16.667	13.691	14.478	14.233	14.333	14.276
3	−16.667	−19.663	−18.666	−19.031	−18.886	−18.949
4	−16.667	−16.471	−16.323	−16.513	−16.373	−16.459
5	0.0000	4.514	2.8830	3.418	3.252	3.304
6	16.667	13.691	14.478	14.233	14.333	14.27
7	16.667	19.442	18.399	18.814	18.639	18.718
8	−16.667	−17.869	−17.455	−17.646	−17.548	−17.6
9	25.000	23.13	22.771	23.223	22.942	23.093
10	−16.667	−15.417	−15.014	−15.537	−15.17	−15.395
Approximate PF						
1	1.90E–03	5.40E–03	5.80E–03	5.80E–03	5.80E–03	5.80E–03
2	1.04E–02	3.10E–03	2.50E–03	2.70E–03	2.60E–03	2.60E–03
3	6.00E–04	6.30E–03	6.70E–03	6.70E–03	6.60E–03	6.70E–03
4	2.60E–03	2.40E–03	2.00E–03	2.20E–03	2.10E–03	2.10E–03
5	2.00E–04	8.00E–04	1.60E–03	1.20E–03	1.40E–03	1.30E–03
6	1.14E–02	3.40E–03	2.70E–03	2.90E–03	2.80E–03	2.80E–03
7	1.60E–03	9.70E–03	1.01E–02	1.01E–02	1.01E–02	1.01E–02
8	1.60E–03	3.70E–03	3.90E–03	4.00E–03	3.90E–03	4.00E–03
9	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
10	1.15E–02	7.00E–03	6.40E–03	6.30E–03	6.50E–03	6.30E–03
System	4.18E–02	4.18E–02	4.17E-02	4.19E–02	4.18E–02	4.17E–02
Actual PF						
1	1.90E–03	8.20E–03	4.90E–03	6.20E–03	5.60E–03	5.90E–03
2	1.04E–02	1.70E–03	3.00E–03	2.50E–03	2.70E–03	2.60E–03
3	6.00E–04	1.28E–02	5.20E–03	7.30E–03	6.40E–03	6.80E–03
4	2.60E–03	2.20E–03	1.90E–03	2.30E–03	2.00E–03	2.20E–03
5	2.00E–04	2.40E–03	1.10E–03	1.50E–03	1.30E–03	1.40E–03
6	1.14E–02	1.90E–03	3.20E–03	2.70E–03	2.90E–03	2.80E–03
7	1.60E–03	1.74E–02	8.00E–03	1.11E–02	9.70E–03	1.03E–02
8	1.60E–03	5.20E–03	3.50E–03	4.20E–03	3.80E–03	4.00E–03
9	0.00	0.00	0.00	0.00	0.00	0.00
10	1.15E–02	7.50E–03	5.00E–03	7.50E–03	5.80E–03	6.80E–03
System	4.18E–02	5.93E–02	3.58E–02	4.53E–02	4.02E–02	4.28E–02

of the areas. Results of the probabilistic optimization are shown in Table 11. The table shows the change in area as well as the stresses corresponding to the mean values of the variables, and the change in probability of failure of each element. It also presents evaluation of system probability of failure using 10 million samples of traditional MCS at the probabilistic optimum design. Overall weight was reduced by 6% (90.47 lb), while maintaining the same bound on the system probability of failure as the deterministic optimum design. We used 10,000 samples for conditional MCS, and optimization problem converged after 59 iterations. A total of 728 reliability assessments were required during the probabilistic optimization. We used fixed random variables to eliminate the effect of random noise.

In the probabilistic optimum design, the areas of three small-area elements (2, 6, and 10) that have high probability of failure in the deterministic optimum are slightly increased and the corresponding stresses are slightly reduced. However, the reduction in the probability of failure is significant for these elements. Thus, the risk reallocation between different members occurs by moving small amount of weight to the small-weight, high-risk members. By reallocating the risk, the total weight is saved by 6%.

4.5 Approximate probabilistic optimization using ECARD

The approximate probabilistic optimization problem can be written based on (21) as

$$\begin{aligned} &\underset{\overline{A}_i}{\text{Minimize}} \;\; W = \sum_{i=1}^{10} \rho L_i \overline{A}_i \\ &\text{s.t.} \qquad P_{FSU}^{\text{ECARD}} \leq P_{FSU}^{\text{det}} \end{aligned} \tag{22}$$

where the approximate system probability of failure is the sum of individual contributions. The optimization in (22) is solved iteratively until convergence criterion in Section 2.2 satisfies. In order to approximate the response of a design when the cross sectional areas change from $\mathbf{A}_{\text{det}}$ to $\mathbf{A}$ we use the following equations:

$$\begin{aligned} r(p_1; p_2; \ldots; \mathbf{A}) &= r(p_1; p_2; \ldots; \mu_{\mathbf{A}}) + \Delta; \\ \Delta &= r(\mu_{p1}; \mu_{p_2}; \ldots; \mu_{\mathbf{A}}) \\ &\quad - r(\mu_{p1}; \mu_{p_2}; \ldots; \mu_{\mathbf{A}\,\text{det}}) \end{aligned} \tag{23}$$

Table 12 shows the results of approximate probabilistic optimization and progress toward the probabilistic optimum shown in Table 11. For this example, the ECARD method converged after five iterations when the changes in design variables and weight were below 0.4% and the accuracy of the system probability of failure was about 5%. In addition, the errors in element failure probability approximations are less than 7%. Since the probability of failure of the Element 9 is very small, the error in its probability of failure is not taken into account. As expected, the mean stresses in the light weight elements decrease, while the mean stresses in the heavier elements increase. This reflects the fact that having higher safety factor to low-weight elements is more weight efficient than to high-weight elements in the overall risk allocation. Note that while the total weight is very similar to the exact probabilistic design, the individual areas are quite different. For example, in the probabilistic design, the stress reduction in Element 2 is achieved by tripling its area, while in the ECARD design the stress is reduced without changing the area. Also notice that the total approximate failure probability bound stays close to 4.2×10^{-2} with small variations due to numerical noise in the algorithm. The true probability bound however changes as design changes.

The ECARD optimization required only five reliability assessments one for each of the five iterations, which is a significant reduction from the probabilistic optimization that consumed 728 reliability assessments. Table 13 shows system probability of failure using 10 million samples at the design obtained in last iteration of ECARD Method. Again, the system failure probability is substantially lower than the Ditlevsen bound. This suggests that it may be worthwhile to apply ECARD in the future with conditional MCS so as to allow the use of a system probability constraint.

Table 13 Probabilities of failure of each element and system probability of failure after last Iteration of ECARD using crude MCS

Elements	A_i	10^7 Samples traditional MCS $(P_f)_i$	10^4 Separable MCS $(P_f)_i$
1	7.395	6.00E–03	5.90E–03
2	0.100	2.70E–03	2.60E–03
3	7.0944	6.90E–03	6.80E–03
4	3.9636	1.80E–03	2.20E–03
5	0.100	1.20E–03	1.40E–03
6	0.100	2.70E–03	2.80E–03
7	5.1198	1.03E–02	1.03E–02
8	5.2688	4.20E–03	4.00E–03
9	3.9953	0.00E+00	0.00
10	0.1311	6.00E–03	6.80E–03
Total	1,414.0 lbs	4.18E–02	4.28E–02
System failure probability		3.33E–02	–

5 Concluding remarks

An approximate probabilistic optimization method using exact-capacity-approximate-response-distribution

(ECARD) is presented. The proposed method significantly reduces expensive reliability calculations (typically done via MCS). ECARD was demonstrated with the simplest approximation of the response distribution—one that translates the entire distribution based on the mean values of the random variables. Two examples were used to test ECARD. First, probabilistic optimization of a cantilever beam was performed, where risk was allocated between two different failure modes. Then, probabilistic optimization of a ten-bar truss problem was performed, where risk was allocated between ten truss members. From the results obtained in these two demonstration problems, we reached the following conclusions.

1. In both problems, ECARD converged to near optima that allocated risk between failure modes much more efficiently than the deterministic optima. The differences between the accurate and approximate optima were due to the errors in probability of failure estimations, which led to errors in the derivatives of probabilities of failure with respect to design variables required for risk allocation.
2. ECARD significantly reduced the expensive reliability calculations. In the cantilever beam problem, it required only four reliability assessments compared to 61 for full probabilistic optimization. Similarly, in the ten-bar truss problem we needed only five reliability assessments, which is almost two-orders of improvement from 728.
3. Large differences between the Ditlevsen bound and the system probability of failure for the 10-bar truss suggest the desirability of applying ECARD with system probability constraints using the separable MCS (Smarslok et al. 2006) instead of the more restrictive conditional MCS.

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Appendix: calculation of member forces of ten-bar truss

Analytical solution to ten-bar truss problem is given in Elishakoff et al. (1994). The member forces satisfy the following equilibrium and compatibility equations.

Note: Values with "*" are incorrect in the reference. The correct expressions are:

$$N_1 = P_2 - \frac{1}{\sqrt{2}}N_8,\ N_2 = -\frac{1}{\sqrt{2}}N_{10},$$

$$N_3 = -P_1 - 2P_2 - \frac{1}{\sqrt{2}}N_8,\ N_4 = -P_2 - \frac{1}{\sqrt{2}}N_{10},$$

$$N_5 = -P_2 - \frac{1}{\sqrt{2}}N_8 - \frac{1}{\sqrt{2}}N_{10},$$

$$N_6 = -\frac{1}{\sqrt{2}}N_{10},\ N_7 = \sqrt{2}\,(P_1 + P_2) + N_8,$$

$$N_8^* = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}},\ N_9 = \sqrt{2}P_2 + N_{10},$$

$$N_{10}^* = \frac{a_{11} b_2 - a_{21} b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

where

$$a_{11}^* = \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right),$$

$$a_{12}^* = a_{21}^* = \frac{1}{A_5},$$

$$a_{22}^* = \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right),$$

$$b_1^* = \sqrt{2}\left(\frac{P_2}{A_1} - \frac{P_1 + 2P_2}{A_3} - \frac{P_2}{A_5} - \frac{2\sqrt{2}\,(P_1 + P_2)}{A_7}\right),$$

$$b_2^* = \left(\frac{-\sqrt{2}P_2}{A_4} - \frac{\sqrt{2}P_2}{A_5} - \frac{4P_2}{A_9}\right)$$

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