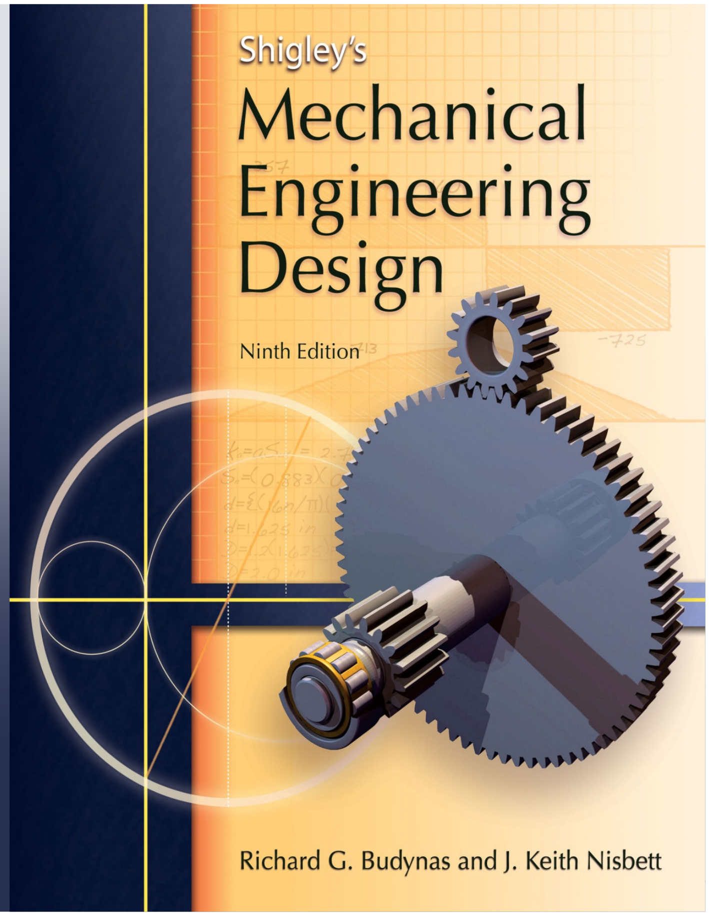
Lecture Slides

Chapter 4

Deflection and Stiffness



Chapter Outline

Stiffness

- Describes the resistance of a structure to deformation in response to an applied load.
  - The inverse of stiffness is called *compliance.*

- **Axially-Loaded Stiffness**
  - $k = F/\delta$, $\quad \delta = \dfrac{Fl}{AE} \quad \longrightarrow \quad \boxed{k = \dfrac{AE}{l}}$

- **Torsionally-Loaded Stiffness**

$$k = \frac{T}{\theta} \qquad \theta = \frac{Tl}{GJ} \quad \longrightarrow \quad \boxed{k = \frac{T}{\theta} = \frac{GJ}{l}}$$

Deflection Due to Bending

- Curvature of beam subjected to bending moment *M*

$$\frac{1}{\rho} = \frac{M}{EI} \qquad (4–8)$$

- From mathematics, curvature of plane curve

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \qquad (4–9)$$

- Slope of beam at any point *x* along the length

$$\theta = \frac{dy}{dx}$$

- If the slope is very small, the denominator of Eq. (4-9) approaches unity.
- Combining Eqs. (4-8) and (4-9), for beams with small slopes,

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

Deflection Due to Bending

- Recall Eqs. (3-3) and (3-4)

$$V = \frac{dM}{dx} \quad (3\text{–}3)$$

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q \quad (3\text{–}4)$$

- Successively differentiating

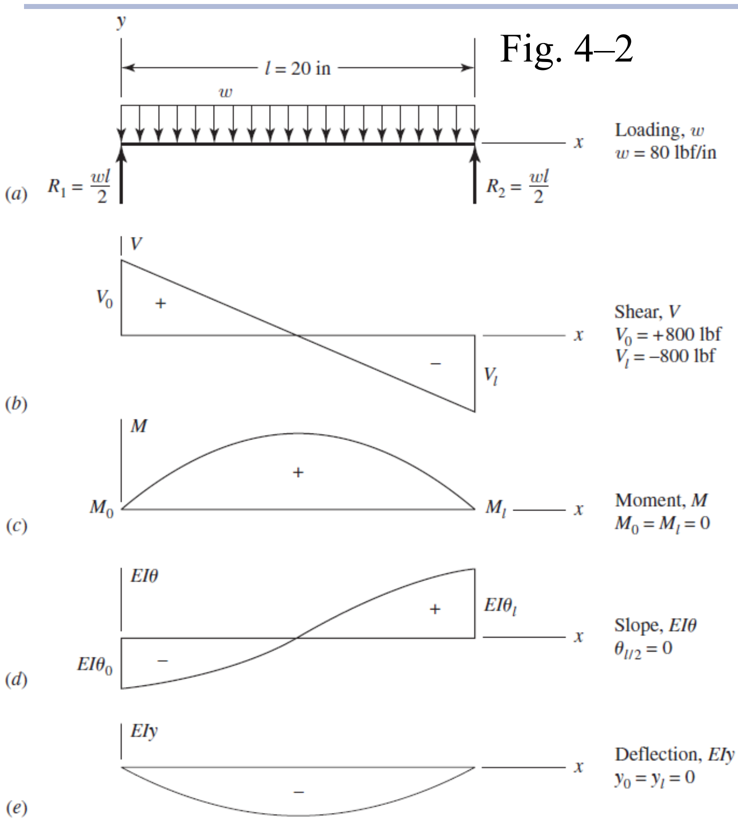
$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

Deflection Due to Bending

Fig. 4–2



$$\frac{q}{EI} = \frac{d^4y}{dx^4} \quad (4\text{-}10)$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3} \quad (4\text{-}11)$$

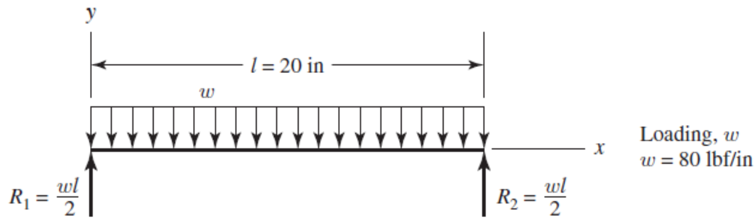
$$\frac{M}{EI} = \frac{d^2y}{dx^2} \quad (4\text{-}12)$$

$$\theta = \frac{dy}{dx} \quad (4\text{-}13)$$

$$y = f(x) \quad (4\text{-}14)$$

Example 4-1

Using Eq. (4–12), determine the equations for the slope and deflection of the beam, the slopes at the ends, and the maximum deflection.



$$\frac{M}{EI} = \frac{d^2y}{dx^2} \qquad (4\text{-}12)$$

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Beam Deflection Methods

- Some of the more common methods for solving the integration problem for beam deflection
  - Superposition
  - Moment-area method
  - Singularity functions
  - Numerical integration
- Other methods that use alternate approaches
  - Castigliano energy method
  - Finite element software

Beam Deflection by Superposition

- *Superposition* determines the effects of each load separately, then adds the results.
- Separate parts are solved using any method for simple load cases.
- **Conditions to use Superposition Method**
  - Each effect is linearly related to the load that produces it.
  - A load does not create a condition that affects the result of another load.
  - The deformations are not large enough to appreciably alter the geometric relations of the parts of the structural system.

Example 4-3

Consider the beam in Fig. 4–4*a* and determine the deflection equations using superposition.

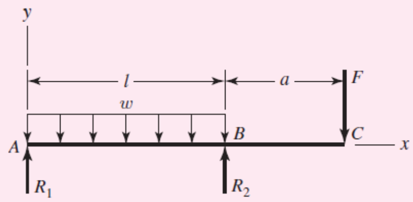

Fig. 4–4 (*a*)

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Strain Energy

- External work done on elastic member in deforming it is transformed into *strain energy*, or *potential energy*.

- Strain energy equals product of average force and deflection.

$$U = \frac{F}{2}y = \frac{F^2}{2k} \qquad (4\text{–}15)$$

- Strain energy density is equal to the area under stress-strain curve.

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Some Common Strain Energy Formulas

- For axial loading, applying $k = AE/l$ from Eq. (4-4),

$$U = \frac{F^2 l}{2AE} \quad (4\text{–}16)$$

or

$$U = \int \frac{F^2}{2AE}\,dx \quad (4\text{–}17)$$

} tension and compression

- For torsional loading, applying $k = GJ/l$ from Eq. (4-7),

$$U = \frac{T^2 l}{2GJ} \quad (4\text{–}18)$$

or

$$U = \int \frac{T^2}{2GJ}\,dx \quad (4\text{–}19)$$

} torsion

Some Common Strain Energy Formulas

- For direct shear loading,

$$U = \frac{F^2 l}{2AG} \quad (4\text{–}20)$$

or

$$U = \int \frac{F^2}{2AG}\,dx \quad (4\text{–}21)$$

} direct shear

- For bending loading,

$$U = \frac{M^2 l}{2EI} \quad (4\text{–}22)$$

or

$$U = \int \frac{M^2}{2EI}\,dx \quad (4\text{–}23)$$

} bending

Some Common Strain Energy Formulas

- For transverse shear loading,

$$U = \frac{CV^2 l}{2AG} \quad (4\text{–}24)$$

or

$$U = \int \frac{CV^2}{2AG}\,dx \quad (4\text{–}25)$$

} transverse shear

where C is a modifier dependent on the cross sectional shape.

Table 4–1

Strain-Energy Correction Factors for Transverse Shear

Source: Richard G. Budynas, *Advanced Strength and Applied Stress Analysis,* 2nd ed., McGraw-Hill, New York, 1999. Copyright © 1999 The McGraw-Hill Companies.

Beam Cross-Sectional Shape	Factor C
Rectangular	1.2
Circular	1.11
Thin-walled tubular, round	2.00
Box sections†	1.00
Structural sections†	1.00

†Use area of web only.

Summary of Common Strain Energy Formulas

$$U = \frac{F^2 l}{2AE}, \quad U = \int \frac{F^2}{2AE}\,dx$$

} tension and compression

$$U = \frac{T^2 l}{2GJ}, \quad U = \int \frac{T^2}{2GJ}\,dx$$

} torsion

$$U = \frac{F^2 l}{2AG}, \quad U = \int \frac{F^2}{2AG}\,dx$$

} direct shear

$$U = \frac{M^2 l}{2EI}, \quad U = \int \frac{M^2}{2EI}\,dx$$

} bending

$$U = \frac{CV^2 l}{2AG}, \quad U = \int \frac{CV^2}{2AG}\,dx$$

} transverse shear

Example 4-8

A cantilever beam with a round cross section has a concentrated load F at the end, as shown in Fig. 4–9a. Find the strain energy in the beam.

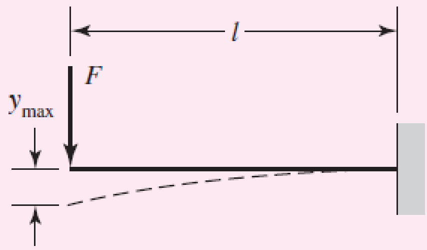

Fig. 4–9 (a)

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Castigliano's Theorem

- When forces act on elastic systems subject to small displacements, the displacement corresponding to any force, in the direction of the force, is equal to the partial derivative of the total strain energy with respect to that force.

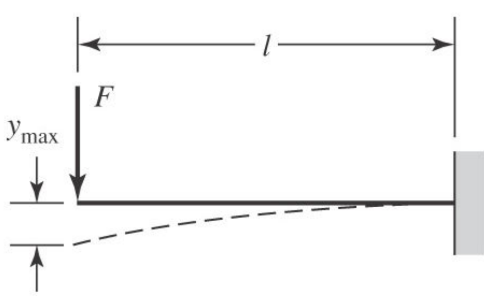
$$\delta_i = \frac{\partial U}{\partial F_i} \qquad (4\text{–}26)$$

- For rotational displacement, in radians,

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad (4\text{–}27)$$

Example 4-9

The cantilever of Ex. 4–8 is a carbon steel bar 10 in long with a 1-in diameter and is loaded by a force $F = 100$ lbf.
(*a*) Find the maximum deflection using Castigliano's theorem, including that due to shear.
(*b*) What error is introduced if shear is neglected?



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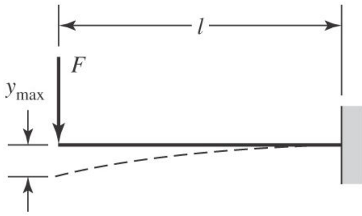
Utilizing a Fictitious Force

- Castigliano's method can be used to find a deflection at a point even if there is no force applied at that point.
- Apply a fictitious force Q at the point, and in the direction, of the desired deflection.
- Set up the equation for total strain energy including the energy due to Q.
- Take the derivative of the total strain energy with respect to Q.
- Once the derivative is taken, Q is no longer needed and can be set to zero.

$$\delta = \left. \frac{\partial U}{\partial Q} \right|_{Q=0} \qquad (4\text{–}28)$$

Example 4-9(b)

Using Castigliano’s method and utilizing a fictitious force, derive an expression for the **mid deflection** of the cantilever beam shown.



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Finding Deflection Without Finding Energy

- For cases requiring integration of strain energy equations, it is more efficient to obtain the deflection directly without explicitly finding the strain energy.
- The partial derivative is moved inside the integral.
- For example, for bending,

$$\delta_i = \frac{\partial U}{\partial F_i} = \frac{\partial}{\partial F_i}\left(\int \frac{M^2}{2EI}\,dx\right) = \int \frac{\partial}{\partial F_i}\left(\frac{M^2}{2EI}\right)dx = \int \frac{2M\frac{\partial M}{\partial F_i}}{2EI}\,dx$$

$$= \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx$$

- Derivative can be taken before integration, simplifying the math.
- Especially helpful with fictitious force Q, since it can be set to zero after the derivative is taken.

Common Deflection Equations

$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE}\left(F\frac{\partial F}{\partial F_i}\right)dx$	tension and compression	(4–29)
$\theta_i = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ}\left(T\frac{\partial T}{\partial M_i}\right)dx$	torsion	(4–30)
$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx$	bending	(4–31)

Example 4-10

Using Castigliano's method, determine the deflections of points A and B due to the force F applied at the end of the step shaft shown in Fig. 4–10. The second area moments for sections AB and BC are I_1 and $2I_1$, respectively.

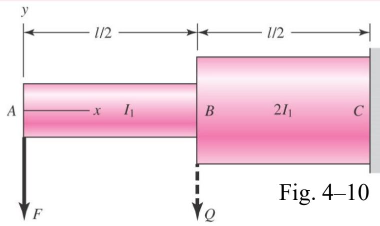

Fig. 4–10

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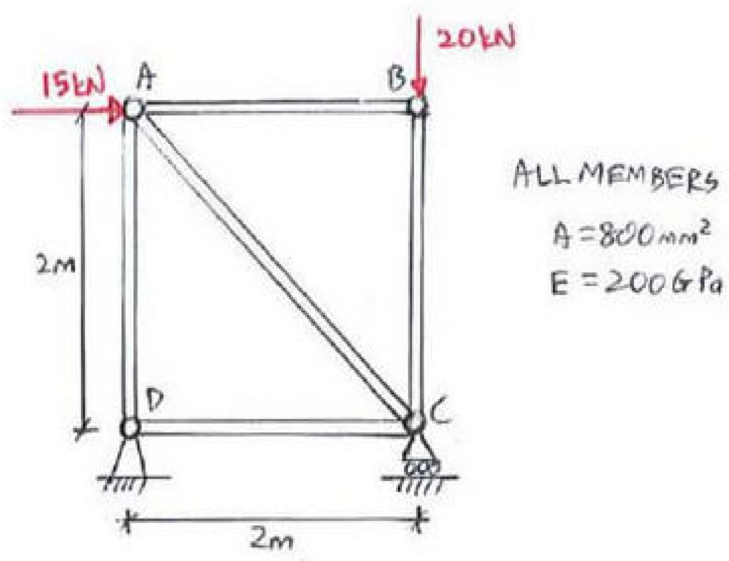
Another Example

Using Castigliano's method, find the horizontal displacement of joint B.



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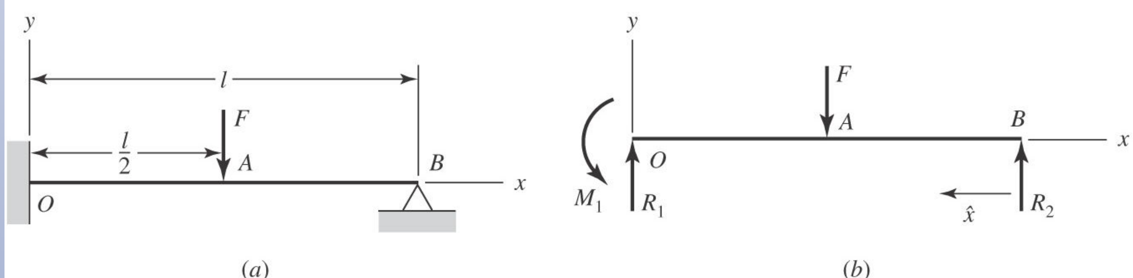
Procedure 1 for Statically Indeterminate Problems

1. Choose the redundant reaction(s)
2. Write the equations of static equilibrium for the remaining reactions in terms of the applied loads and the redundant reaction(s).
3. Write the deflection equation(s) for the point(s) at the locations of the redundant reaction(s) in terms of the applied loads and redundant reaction(s).
4. Solve equilibrium equations and deflection equations simultaneously to determine the reactions.

Note: Procedure 2 will not be covered in this class, but you may want to take a look at it.

Example 4-14

Determine the reactions using procedure 1.



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Compression Members

- *Column* – A member loaded in compression such that either its length or eccentric loading causes it to experience more than pure compression

- **Four categories of columns**
  - Long columns with central loading
  - Short columns with central loading
  - Long columns with eccentric loading
  - Short columns with eccentric loading

Euler Column Formula

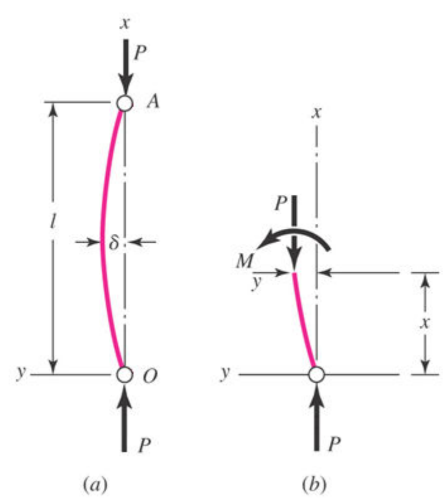
- Internal moment: $M = -P\,y$

- Substituting into $d^2y/dx^2 = M/EI$,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

- Solving with boundary conditions $y = 0$ at $x = 0$ and at $x = l$

$$\sqrt{\frac{P}{EI}}\,L = n\pi \quad (n = 1, 2, \cdots)$$



- The smallest critical load is

$$P_{cr} = \frac{\pi^2 EI}{l^2} \qquad \textbf{(\textit{Euler buckling load})}$$

Solution of the differential equation

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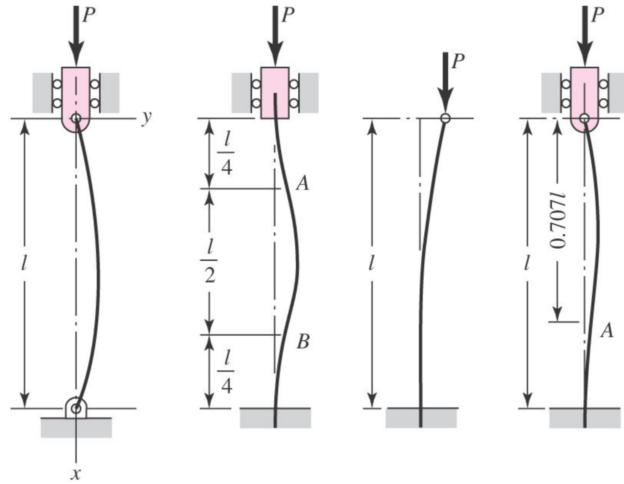
Long Columns with Central Loading

- When P reaches *critical load*, column becomes *unstable* and bending develops rapidly.
- Critical load depends on end conditions (***Euler column formula***)

$$P_{cr} = \frac{C\pi^2 EI}{l^2}$$

- More conservative values of C are often used.



(a) $C = 1$ (b) $C = 4$ (c) $C = \frac{1}{4}$ (d) $C = 2$

Column End Conditions	End-Condition Constant C: Theoretical Value	Conservative Value	Recommended Value*
Fixed-free	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Rounded-rounded	1	1	1
Fixed-rounded	2	1	1.2
Fixed-fixed	4	1	1.2

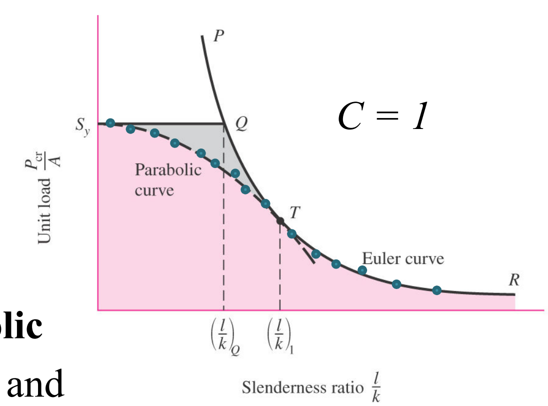
*To be used only with liberal factors of safety when the column load is accurately known.

Comparison with Test Results

- Using $I = Ak^2$, where A is the area and k is the radius of gyration, Euler column formula can be expressed as

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2} \qquad (4\text{-}44)$$

- l/k : *slenderness ratio,*
- P_{cr}/A : *critical unit load*



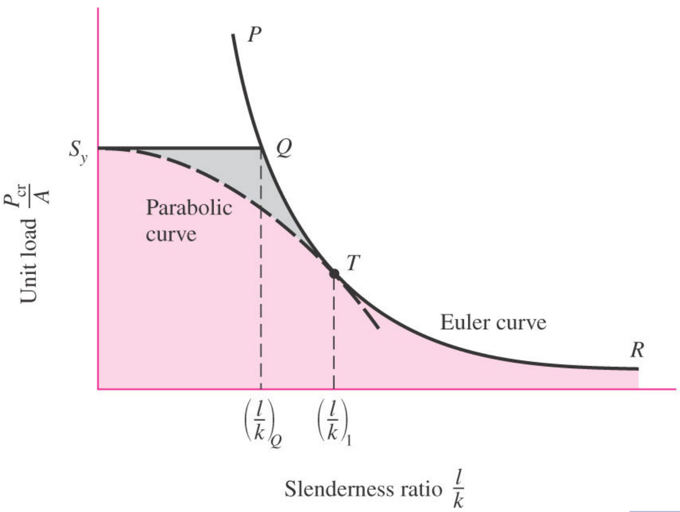
- Test results indicate that a **parabolic curve** can be used before point T, and **Euler curve** can be used after point T.

Point T is usually defined such that $P_{cr}/A = S_y/2$, giving

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2}$$

Condition for Use of Euler Equation

- For **long columns**, where $(l/k) > (l/k)_1$,
  - use Euler equation
- For **intermediate-length columns**, where $(l/k) \leq (l/k)_1$,
  - use a parabolic curve between S_y and T



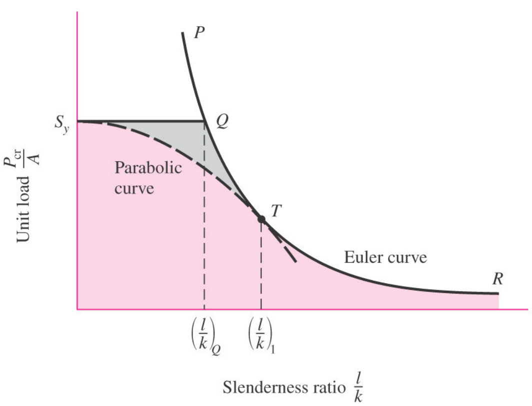
Short Columns with Central Loading

- General form of parabola

$$\frac{P_{cr}}{A} = a - b\left(\frac{l}{k}\right)^2$$

- Parabola starts at S_y, so $a = S_y$
- Parabola is tangent to Euler curve at T, so

$$b = \left(\frac{S_y}{2\pi}\right)^2 \frac{1}{CE}$$



- Also known as *J.B. Johnson formula*

$$\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{l}{k}\right)^2 \frac{1}{CE} \qquad \frac{l}{k} \leq \left(\frac{l}{k}\right)_1$$

Example 4-17

Specify the diameter of a round column 1.5 m long that is to carry a maximum load estimated to be 22 kN. Use a design factor $n_d = 4$ and consider the ends as pinned (rounded). The column material selected has a minimum yield strength of 500 MPa and a modulus of elasticity of 207 GPa.

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Columns with Eccentric Loading

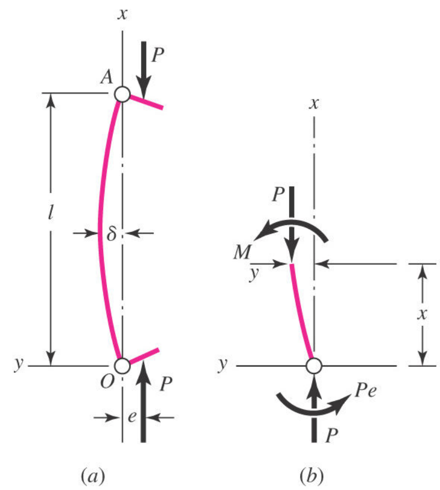
- For eccentrically loaded column with eccentricity e,

$$M = -P(e+y)$$

- Substituting into $d^2y/dx^2=M/EI$,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}$$

- Solving with boundary conditions $y = 0$ at $x = 0$ and at $x = l$



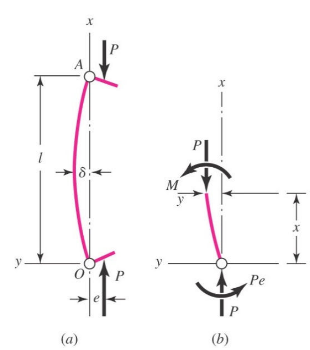
$$y = e\left[\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1\right]$$

Columns with Eccentric Loading

- At midspan where $x = l/2$

$$\delta = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right) - 1\right]$$

$$M_{\max} = P(e+\delta) = Pe\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$



- The maximum compressive stress includes axial and bending

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ak^2}$$

- Substituting $M_{\max}$, we get

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{l}{2k}\sqrt{\frac{P}{EA}}\right)\right]$$

- Using S_{yc} as the maximum value of σ_c, and solving for P/A, we obtain the *secant column formula*

$$\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2)\sec[(l/2k)\sqrt{P/AE}]}$$

Secant Column Formula

- Secant Column Formula

$$\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2)\sec[(l/2k)\sqrt{P/AE}]} \qquad (4\text{–}50)$$

- ec/k^2 is the *eccentricity ratio*
- Design charts of secant column formula for various eccentricity ratio can be prepared for a given material strength

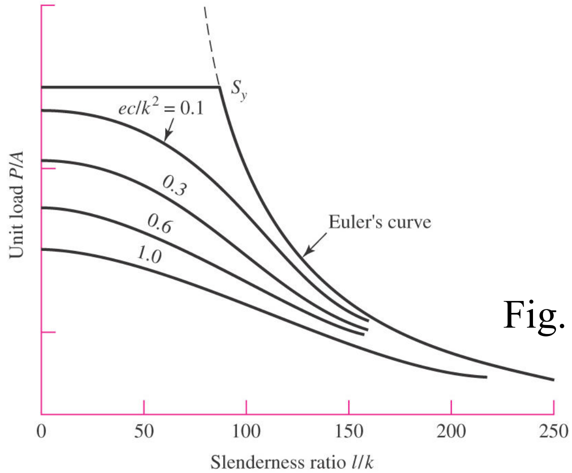


Fig. 4–21

Short Columns Under Eccentric Loading

- If eccentricity exists, maximum stress is at *B* with axial compression and bending.

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{PecA}{IA} = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right) \qquad (4\text{–}55)$$

- Notice that it is not a function of length
- Differs from secant equation in that it assumes small effect of bending deflection
- If bending deflection is limited to 1 percent of *e*, then from Eq. (4-44), the limiting slenderness ratio for short columns under eccentric loading is

$$\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2} \qquad (4\text{–}56)$$

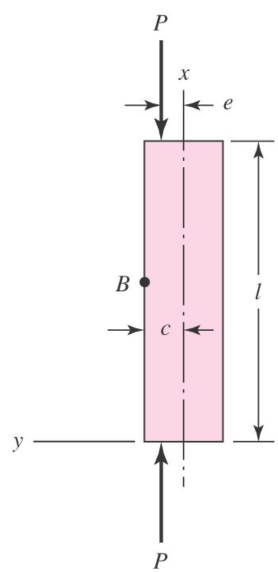


Fig. 4–22

Example 4-20

Figure 4–23*a* shows a workpiece clamped to a milling machine table by a bolt tightened to a tension of 2000 lbf. The clamp contact is offset from the centroidal axis of the strut by a distance $e = 0.10$ in, as shown in part *b* of the figure. The strut, or block, is steel, 1 in square and 4 in long, as shown. Determine the maximum compressive stress in the block.

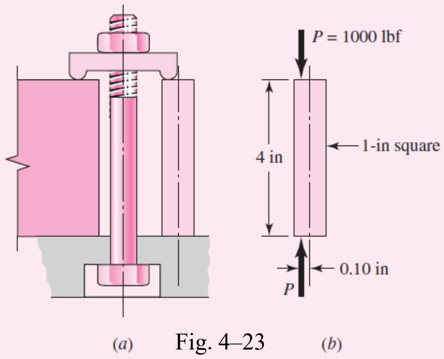


Fig. 4–23

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